

# EXPONENTIAL TREND TO EQUILIBRIUM FOR THE INELASTIC BOLTZMANN EQUATION DRIVEN BY A PARTICLE BATH

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**ABSTRACT.** We consider the spatially homogeneous Boltzmann equation for inelastic hard spheres (with constant restitution coefficient  $\alpha \in (0, 1)$ ) under the thermalization induced by a host medium with a fixed Maxwellian distribution. We prove that the solution to the associated initial-value problem converges exponentially fast towards the unique equilibrium solution. The proof combines a careful spectral analysis of the linearised semigroup as well as entropy estimates. The trend towards equilibrium holds in the weakly inelastic regime in which  $\alpha$  is close to 1, and the rate of convergence is explicit and depends solely on the spectral gap of the *elastic* linearised collision operator.

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## 1. INTRODUCTION

We pursue our investigation initiated in [3, 4] of the qualitative properties of inelastic hard spheres suspended in a thermal medium. In a more precise way, we investigate here the large time behavior of the one-particle distribution function  $f(v, t)$ ,  $v \in \mathbb{R}^3$ ,  $t > 0$  solution to the following spatially homogeneous Boltzmann equation:

$$\partial_t f = \mathcal{Q}_\alpha(f, f) + \mathbf{L}(f), \quad (1.1)$$

where  $\mathcal{Q}_\alpha(f, f)$  is the inelastic quadratic Boltzmann collision operator, while  $\mathbf{L}(f)$  models the forcing term. The parameter  $\alpha$  is the *restitution coefficient*, expressing the degree of inelasticity of binary collisions between grains:  $0 < \alpha \leq 1$ , and the purely elastic case is recovered when  $\alpha = 1$ .

**1.1. Setting of the problem.** As is well documented, dilute granular flows can be described by kinetic models associated to suitable modifications of the Boltzmann operator for which hard-sphere collisions are assumed to be inelastic [6]: each encounter dissipates a fraction of the kinetic energy. In absence of energy supply the system cools down and the corresponding dissipative Boltzmann equation admits only trivial equilibria. This is no longer the case if the spheres are forced to interact with an external thermostat, in which case the energy supply may lead to a non trivial steady state. Different kinds of forcing term have been considered in the literature [10, 21, 15, 16, 17, 11], and the qualitative properties of the corresponding steady states have been investigated. In particular, the following questions have been addressed regarding steady solutions of kinetic equations of the type (1.1): (1) Existence [15, 11], (2) Uniqueness in some weakly inelastic regime corresponding to  $\alpha$  close to 1 [16, 17] and (3) stability, i.e. convergence of the solution to the associated Boltzmann equation towards the steady state [16, 17]. In particular, for hard spheres subject to diffuse forcing, the large-time behaviour of the solution to the BE has been completely characterised in [17] whereas, for anti-drift forcing (closely related to self-similar solutions to the freely evolving Boltzmann equation), asymptotic behaviour has been considered in [16].

In this paper we are concerned with the asymptotic behaviour of a physical model in which the system of inelastic hard spheres is immersed in a thermal bath of particles at equilibrium, already investigated in [3, 4]. In this model the forcing term is given by a linear scattering operator describing *elastic collisions* with the background medium:

$$\mathbf{L}(f) = \mathcal{Q}_1(f, \mathcal{M})$$

where  $\mathcal{M}$  stands for the distribution function of the host fluid, supposed to be a Maxwellian with unit mass, bulk velocity  $u_0 \in \mathbb{R}^3$  and temperature  $\Theta_0 > 0$ :

$$\mathcal{M}(v) = \left( \frac{1}{2\pi\Theta_0} \right)^{3/2} \exp \left\{ -\frac{(v - u_0)^2}{2\Theta_0} \right\}, \quad v \in \mathbb{R}^3. \quad (1.2)$$

The precise definitions of the collision operators  $\mathcal{Q}_\alpha(f, f)$  and  $\mathbf{L}(f)$  are given in Subsection 2.1. We refer to [15, 16, 17, 11] for a mathematical discussion of various models and their physical motivation. We restrict our attention to interacting hard spheres and refer

to [7] for exhaustive references on the pseudo-Maxwell approximation. A salient feature of both collision operators  $\mathcal{Q}_\alpha(f, f)$  and  $\mathbf{L}$  is that their only collision invariant is mass, i.e.

$$\int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f) \, dv = \int_{\mathbb{R}^3} \mathbf{L}f \, dv = 0.$$

In contrast with the elastic Boltzmann operator, neither the momentum nor the energy are conserved by  $\mathcal{Q}_\alpha$  or  $\mathbf{L}$ .

The existence of smooth stationary solutions for the inelastic Boltzmann equation under the above thermalization has already been proved in [3], for any choice of the restitution coefficient  $\alpha$ . This has been achieved by controlling the  $L^p$ -norms, the moments and the regularity of the solutions for the Cauchy problem, together with a dynamical argument based on the Tychonoff fixed-point theorem.

Uniqueness of the steady state is proven in some previous contribution [4] for a smaller range of parameters  $\alpha$ . Namely, the main results of both [3] and [4] can be summarized as follows:

**Theorem 1.1** (Existence and Uniqueness of the steady state). *For any  $\varrho \geq 0$  and  $\alpha \in (0, 1]$ , there exists a steady solution  $F_\alpha \in L^1_2(\mathbb{R}^3)$ ,  $F_\alpha(v) \geq 0$  to the problem*

$$\mathcal{Q}_\alpha(F_\alpha, F_\alpha) + \mathbf{L}(F_\alpha) = 0 \tag{1.3}$$

with  $\int_{\mathbb{R}^3} F_\alpha(v) \, dv = \varrho$ .

Moreover, there exists  $\alpha_0 \in (0, 1]$  such that such a solution is unique for  $\alpha \in (\alpha_0, 1]$ . The (unique) steady state  $F_\alpha$  for  $\alpha \in (\alpha_0, 1]$  is radially symmetric and belongs to  $C^\infty(\mathbb{R}^3)$ .

We assume in the sequel that  $\alpha \in (\alpha_0, 1]$  and we are interested in the large-time behaviour of the solution  $f(t, v)$  to (1.1) (whose existence and uniqueness are guaranteed by [3]). In particular, if one assumes

$$\int_{\mathbb{R}^3} f_0(v) \, dv = 1, \tag{1.4}$$

then one expects the solution  $f(t, v)$  to converge towards the unique steady solution with unit mass given in Theorem 1.1. Our goal in the present paper is to provide sufficient conditions on the initial datum  $f_0$  ensuring that this convergence holds true, with an exponential rate that we make explicit. As in [16, 17], this exponential trend to equilibrium is proven to hold in the weakly inelastic regime, i.e. for a range of parameters  $\alpha \in (\alpha^\dagger, 1]$  for a certain explicit  $\alpha^\dagger > \alpha_0$ .

**1.2. Main result and strategy of proof.** Our goal is to prove a quantitative version of the return to equilibrium for the solution to (1.1). Our main result combines local stability estimates in a certain weighted  $L^1$ -space with suitable entropy estimates. A crucial point in our approach is that it strongly relies on the understanding of the elastic problem corresponding to  $\alpha = 1$ . In the elastic case, as is well known,  $F_1 = \mathcal{M}$  is exactly the host-medium Maxwellian appearing in  $\mathbf{L}$ . The spectral properties of the linearised operator

around  $\mathcal{M}$  have been studied in [4]. Namely, in the weighted space

$$\mathcal{X} = L^1(\mathbb{R}^3, \exp(a|v|) dv), \quad a > 0,$$

the elastic linearised operator  $\mathcal{L}_1$  given by

$$\mathcal{L}_1 h = \mathcal{Q}_1(h, \mathcal{M}) + \mathcal{Q}_1(\mathcal{M}, h) + \mathbf{L}h, \quad h \in L^1(\mathbb{R}^3; (1 + |v|) \exp(a|v|) dv)$$

admits a positive spectral gap  $\nu > 0$  which can be explicitly estimated. We shall consider solutions to (1.1) associated to a nonnegative initial datum  $f_0 \in \mathcal{X}$  satisfying (1.4) and

$$H(f_0|\mathcal{M}) = \int_{\mathbb{R}^3} f_0(v) \log \left( \frac{f_0(v)}{\mathcal{M}(v)} \right) dv < \infty. \quad (1.5)$$

Our main result can be stated as follows:

**Theorem 1.2.** *For any  $0 < \nu_* < \nu$  (with  $\nu$  equal to the size of the spectral gap of  $\mathcal{L}_1$ ) there exists some explicit  $\alpha^\dagger \in (0, 1)$  such that, for any  $\alpha \in (\alpha^\dagger, 1]$  and any nonnegative initial datum with  $f_0 \in \mathcal{X}$  satisfying (1.4) and (1.5) the solution  $f(t, v)$  to (1.1) satisfies*

$$\|f(t) - F_\alpha\|_{\mathcal{X}} \leq K \exp(-\nu_* t) \quad \forall t \geq 0$$

for some positive constant  $K$  depending on  $\varepsilon, \alpha$  and  $H(f_0|\mathcal{M})$ .

Notice that the rate of convergence is explicitly computable in terms of the spectral gap of the elastic linearised operator  $\mathcal{L}_1$  in  $\mathcal{X}$ .

The proof of the above is based upon two main ingredients:

- (1) A local stability estimate in which exponential convergence is established for small perturbations of the equilibrium state, i.e. whenever the initial datum  $f_0$  is close enough to  $F_\alpha$ .
- (2) Suitable entropy estimates as a tool to pass from local to global stability. We take advantage of the fact that the scattering operator  $\mathbf{L}$  is dominant in the weakly inelastic regime.

To tackle the above first point (1), we have to perform a fine study of the spectral properties of both the linearised operator  $\mathcal{L}_\alpha$  around the steady solution  $F_\alpha$  and its associated evolution semigroup. More precisely, introduce

$$\mathcal{L}_\alpha h = \mathcal{Q}_\alpha(h, F_\alpha) + \mathcal{Q}_\alpha(F_\alpha, h) + \mathbf{L}h, \quad h \in L^1(\mathbb{R}^3; (1 + |v|) \exp(a|v|) dv), \alpha \in (\alpha_0, 1].$$

We deduce the spectral properties of  $\mathcal{L}_\alpha$  in  $\mathcal{X}$  from those of the elastic operator  $\mathcal{L}_1$  by a perturbation argument valid for  $\alpha$  close enough to 1. Notice that the elastic limit  $\alpha \rightarrow 1$  is actually well behaved since the operator gap (in the sense of [13]; see Appendix A) between  $\mathcal{L}_\alpha$  and  $\mathcal{L}_1$  is going to 0 as  $\alpha \rightarrow 1$ . This allows us to apply results from the perturbation theory of unbounded operators [13]. This strongly contrasts with the analysis of [16, 17] which, though perturbative, was ill-behaved in the elastic limit.

As is well known, the spectral properties of the  $C_0$ -semigroup  $(\mathcal{S}_\alpha(t))_{t \geq 0}$  generated by  $\mathcal{L}_\alpha$  cannot be directly deduced from those of  $\mathcal{L}_\alpha$  because of the lack of spectral mapping theorem in infinite dimensional Banach spaces. In particular, one cannot directly derive from the existence of a spectral gap for  $\mathcal{L}_\alpha$  the decay of the associated semigroup. However, following an operator splitting strategy introduced in [18], we can localise the

essential spectrum of  $(\mathcal{S}_\alpha(t))_{t \geq 0}$  through a weak compactness argument and deduce from that the local stability theorem (see Theorem 3.9 and Theorem 4.2). We give a direct and elementary proof that does not rely on the recent results of [12, 18].

In order to address the above point (2) entropy estimates play a crucial role. Our method is based upon the following entropy-entropy production estimate recently obtained in [5]. Introducing the entropy production associated to  $\mathbf{L}$ ,

$$\mathbf{D}(f) = - \int_{\mathbb{R}^3} \mathbf{L}f(v) \log \left( \frac{f(v)}{\mathcal{M}(v)} \right) dv,$$

the result reads as follows:

**Theorem 1.3.** *There exists  $\lambda > 0$  such that*

$$\mathbf{D}(f) \geq \lambda H(f|\mathcal{M}) := \lambda \int_{\mathbb{R}^3} f(v) \log \left( \frac{f(v)}{\mathcal{M}(v)} \right) dv \geq 0$$

*holds for any probability distribution  $f \in L^1(\mathbb{R}^3, dv)$ .*

This result has important consequences on the asymptotic behaviour of the solution to (1.1) in the elastic case  $\alpha = 1$ . In this case the scattering operator  $\mathbf{L}$  becomes dominant and forces the solution  $f(t, v)$  to (1.1) to converge exponentially fast towards the unique equilibrium state of  $\mathbf{L}$ , which is the Maxwellian  $\mathcal{M}$  (see [5] for details). Roughly speaking, in the elastic limit  $\alpha \rightarrow 1$ , we expect the persistence of this behaviour and we expect the scattering operator  $\mathbf{L}$  to drive the system in some neighbourhood of  $\mathcal{M}$ . Since  $F_\alpha \simeq \mathcal{M}$  for  $\alpha$  close to 1, the dynamics is forced to take the solution close to  $\mathcal{M}$  as  $t \rightarrow \infty$ . To be more precise, using the above Theorem 1.3, one can estimate the evolution of the relative entropy along the solutions to (1.1) (see Proposition 5.1) to get

$$H(f(t)|\mathcal{M}) \leq \exp(-\lambda t) H(f_0|\mathcal{M}) + K(1 - \alpha) \quad \forall t \geq 0; \alpha \in (\alpha_0, 1]$$

for some positive constant  $K > 0$  independent of  $\alpha$ . The above estimate ensures that, for large time and  $\alpha \simeq 1$ , the solution to (1.1) will become close enough to  $\mathcal{M}$  and, hence to  $F_\alpha$  which, combined with the local stability theorem, yields our main result. It is worth mentioning here that, while the analysis in [4] dealt with a (possibly) inelastic scattering operator, we restrict ourselves here to *elastic* interactions between the hard spheres and the host medium due to the inavailability of Theorem 1.3 in the inelastic case. Notice however that all the spectral results of the paper, as well as the local stability theorem 4.2, hold true without modification substituting  $\mathbf{L}$  by the inelastic scattering operator

$$\mathbf{L}_e f = \mathcal{Q}_e(f, \mathcal{M}),$$

associated to a general constant restitution coefficient  $e \in (0, 1]$ .

**1.3. Plan of the paper.** The organisation of the paper is as follows. In the next section we define the collision operators  $\mathcal{Q}_\alpha$  and  $\mathbf{L}$ , and recall from [4] the main properties of the solution to (1.1) and the steady state  $F_\alpha$ . Section 3 is devoted to a study of the spectral properties of  $\mathcal{L}_\alpha$  and  $(\mathcal{S}_\alpha(t))_{t \geq 0}$ , used later in Section 4 to derive the local stability Theorem 4.2. In Section 5 we exploit the entropy estimates to establish our main global

stability result. The proof that, for any  $\alpha \in (\alpha_0, 1]$ ,  $\mathcal{L}_\alpha$  generates a  $C_0$ -semigroup in  $\mathcal{X}$  is postponed to Appendix A and uses a weak compactness argument.

**1.4. Notation.** Given two Banach spaces  $X$  and  $Y$ , we denote by  $\mathcal{B}(X, Y)$  the set of linear bounded operators from  $X$  to  $Y$  and by  $\|\cdot\|_{\mathcal{B}(X, Y)}$  the associated operator norm. If  $X = Y$ , we simply denote  $\mathcal{B}(X) := \mathcal{B}(X, X)$ . We denote then by  $\mathcal{C}(X)$  the set of closed, densely defined linear operators on  $X$  and by  $\mathcal{K}(X)$  the set of all compact operators in  $X$ . For  $A \in \mathcal{C}(X)$ , we write  $\mathcal{D}(A) \subset X$  for the domain of  $A$ ,  $\mathcal{N}(A)$  for the null space of  $A$  and  $\text{Range}(A) \subset X$  for the range of  $A$ . The spectrum of  $A$  is then denoted by  $\mathfrak{S}(A)$  and the resolvent set is  $\rho(A)$ . For  $\lambda \in \rho(A)$ ,  $R(\lambda, A)$  denotes the resolvent of  $A$ . We also define the discrete spectrum  $\mathfrak{S}_d(A)$  as the set of eigenvalues of  $A$  with finite algebraic multiplicity (see [13, 9] for more details). We denote by  $s(A)$  the *spectral bound* of  $A$ , i.e.

$$s(A) = \sup\{\text{Re } \lambda; \lambda \in \mathfrak{S}(A)\}.$$

There are several definitions of the essential spectrum of  $A$  in the literature which are unfortunately not equivalent. In the present paper we adopt the notion of *Schechter essential spectrum*, denoted by  $\mathfrak{S}_{\text{ess}}(A)$  and defined by

$$\mathfrak{S}_{\text{ess}}(A) = \bigcap_{K \in \mathcal{K}(X)} \mathfrak{S}(A + K).$$

For a bounded operator  $T \in \mathcal{B}(X)$  we can also define the *essential radius* of  $T$  as

$$\begin{aligned} r_{\text{ess}}(T) &= \inf \{r > 0; \mathfrak{S}(T) \cap \{\lambda \in \mathbb{C}; |\lambda| > r\} \subset \mathfrak{S}_d(T)\} \\ &= \sup \{|\mu|; \mu \in \mathfrak{S}_{\text{ess}}(T)\}. \end{aligned}$$

Notice that the first identity is peculiar to Schechter essential spectrum whereas the second one is valid for any of the various notions of essential spectrum (see [8, Corollary 4.11, p. 44]). If  $(U(t))_{t \geq 0}$  is a  $C_0$ -semigroup in  $X$  with generator  $A$ , we denote by  $\omega_0(U)$  its *growth bound* and by  $\omega_{\text{ess}}(U)$  its *essential type*, defined by

$$\begin{aligned} \exp(t\omega_0(U)) &= \sup \{|\mu|; \mu \in \mathfrak{S}(U(t))\}, \\ \exp(t\omega_{\text{ess}}(U)) &= r_{\text{ess}}(U(t)) = \sup \{|\mu|; \mu \in \mathfrak{S}_{\text{ess}}(U(t))\} \quad \text{for } t \geq 0. \end{aligned}$$

## 2. PRELIMINARY RESULTS

**2.1. The kinetic model.** Given a constant restitution coefficient  $\alpha \in (0, 1)$ , one defines the bilinear Boltzman operator  $\mathcal{Q}_\alpha$  for inelastic interactions and hard spheres by its action on test functions  $\psi(v)$ :

$$\int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, g)(v) \psi(v) dv = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} f(v)g(w) |v - w| (\psi(v') - \psi(v)) dv dw d\sigma \quad (2.1)$$

with  $v' = v + \frac{1+\alpha}{4}(|v - w|\sigma - v + w)$ . In particular, for any test function  $\psi = \psi(v)$ , one has the following weak form of the *quadratic* collision operator:

$$\int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f)(v) \psi(v) dv = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) f(w) |v - w| \mathcal{A}_\alpha[\psi](v, w) dw dv, \quad (2.2)$$

where

$$\begin{aligned}\mathcal{A}_\alpha[\psi](v, w) &= \frac{1}{4\pi} \int_{\mathbb{S}^2} (\psi(v') + \psi(w') - \psi(v) - \psi(w)) \, d\sigma \\ &= \mathcal{A}_\alpha^+[\psi](v, w) - \mathcal{A}_\alpha^-[\psi](v, w)\end{aligned}\quad (2.3)$$

(where we have used the symmetry of the integral under interchange of  $v$  and  $w$ ) and the post-collisional velocities  $(v', w')$  are given by

$$v' = v + \frac{1+\alpha}{4} (|q|\sigma - q), \quad w' = w - \frac{1+\alpha}{4} (|q|\sigma - q), \quad q = v - w. \quad (2.4)$$

In the same way, one defines the linear scattering operator  $\mathbf{L}$  by its action on test functions:

$$\int_{\mathbb{R}^3} \mathbf{L}(f)(v) \psi(v) \, dv = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(v) \mathcal{M}(w) |v - w| \mathcal{J}[\psi](v, w) \, dw \, dv, \quad (2.5)$$

where

$$\mathcal{J}[\psi](v, w) = \frac{1}{4\pi} \int_{\mathbb{S}^2} (\psi(v^\star) - \psi(v)) \, d\sigma = \mathcal{J}^+[\psi](v, w) - \mathcal{J}^-[\psi](v, w). \quad (2.6)$$

with post-collisional velocities  $(v^\star, w^\star)$

$$v^\star = v + \frac{1}{2} (|q|\sigma - q), \quad w^\star = w - \frac{1}{2} (|q|\sigma - q), \quad q = v - w. \quad (2.7)$$

For simplicity, we shall assume in the paper that the particles governed by  $f$  and those with distribution function  $\mathcal{M}$  share the same mass. Notice that

$$\mathbf{L}(f) = \mathcal{Q}_1(f, \mathcal{M})$$

and we shall adopt the convention that post (or pre-) collisional velocities associated to the coefficient  $\alpha$  are denoted with a prime, while those associated to elastic collision are denoted with a  $\star$ . We are interested in the large time behaviour of solutions to the following Boltzmann equation:

$$\partial_t f(t, v) = \mathcal{Q}_\alpha(f(t, \cdot); f(t, \cdot))(v) + \mathbf{L}(f)(t, v), \quad f(0, v) = f_0(v), \quad t > 0, v \in \mathbb{R}^3. \quad (2.8)$$

Notice that

$$\mathcal{Q}_\alpha(f, f) = \mathcal{Q}_\alpha^+(f, f) - \mathcal{Q}_\alpha^-(f, f) = \mathcal{Q}_\alpha^+(f, f) - f \Sigma(f)$$

where

$$\Sigma(f)(v) = (f * |\cdot|)(v) = \int_{\mathbb{R}^3} f(w) |v - w| \, dw.$$

Notice that  $\Sigma(f)$  does not depend on the restitution coefficient  $\alpha \in (0, 1]$ . In the same way,

$$\mathbf{L}(f)(v) = \mathbf{L}^+(f)(v) - \mathbf{L}^-(f)(v) = \mathbf{L}^+(f)(v) - \Sigma(\mathcal{M})(v) f(v)$$

Existence and uniqueness of solutions to (2.8) have been established in [3]. In particular, if  $f_0$  is a nonnegative initial datum with

$$\int_{\mathbb{R}^3} f_0(v) |v|^3 \, dv < \infty \quad \text{and} \quad \int_{\mathbb{R}^3} f_0(v) \, dv = 1 \quad (2.9)$$

then there exists a unique nonnegative solution  $(f(t, v))_{t \geq 0}$  to (2.8) which additionally satisfies

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} f(t, v) |v|^3 dv < \infty, \text{ and } \int_{\mathbb{R}^3} f(t, v) dv = 1 \quad \forall t \geq 0.$$

More generally, uniform propagation of moments holds: namely, for any  $p \geq 2$  one has

$$\int_{\mathbb{R}^3} f_0(v) |v|^p dv < \infty \implies \sup_{t \geq 0} \int_{\mathbb{R}^3} f(t, v) |v|^p dv < \infty. \quad (2.10)$$

See [3, Proposition 4.2 & Theorem 4.8] for more details. Owing to the above mass conservation property we shall restrict ourselves in the sequel to nonnegative initial data satisfying (2.9). Due to the influence of the scattering operator  $\mathbf{L}$  there is no additional conservation law besides mass conservation. In fact, it appears impossible to express the evolution of the momentum

$$\mathbf{u}(t) = \int_{\mathbb{R}^3} f(t, v) v dv \in \mathbb{R}^3$$

and the energy

$$E(t) = \int_{\mathbb{R}^3} f(t, v) |v|^2 dv$$

in a closed form.

**2.2. A posteriori estimates.** We collect here several results obtained in our previous contribution [4] regarding the properties of solutions to (2.8) as well as those of the steady solution  $F_\alpha$  to (1.3). We begin with *high-energy tails* for the solution to (1.1) and  $F_\alpha$ .

**Theorem 2.1.** *Let  $f_0$  be a nonnegative velocity distribution with  $\int_{\mathbb{R}^3} f_0(v) dv = 1$ . Assume that  $f_0$  has an exponential tail of order  $s \in (0, 2]$ , i.e. there exists  $r_0 > 0$  and  $s \in (0, 2]$  such that*

$$\int_{\mathbb{R}^3} f_0(v) \exp(r_0 |v|^s) dv < \infty.$$

*Then there exist  $0 < r \leq r_0$  and  $C > 0$  (independent of  $\alpha \in (0, 1]$ ) such that the solution  $(f(t, v))_{t \geq 0}$  to the Boltzmann equation (2.8) satisfies*

$$\sup_{t \geq 0} \int_{\mathbb{R}^3} f(t, v) \exp(r |v|^s) dv \leq C < \infty. \quad (2.11)$$

*In particular, there exist constants  $A > 0$  and  $M > 0$  such that for all  $\alpha \in (0, 1]$  and all solutions  $F_\alpha$  to (1.3) one has*

$$\int_{\mathbb{R}^3} F_\alpha(v) \exp(A |v|^2) dv \leq M.$$

Notice that the above integral tail estimate for  $F_\alpha$  can actually be strengthened to get the following pointwise Maxwellian bounds:

**Proposition 2.2** ([4, Theorems 4.4 & 4.7]). *There exist two Maxwellian distributions  $\underline{\mathcal{M}}$  and  $\overline{\mathcal{M}}$  (independent of  $\alpha$ ) such that*

$$\underline{\mathcal{M}}(v) \leq F_\alpha(v) \leq \overline{\mathcal{M}}(v) \quad \forall v \in \mathbb{R}^3, \quad \forall \alpha \in (\alpha_0, 1).$$



**2.3. Convergence of  $F_\alpha$  to  $\mathcal{M}$ .** Let us now introduce

$$\mathcal{X} = L^1(m^{-1}) = L^1(\mathbb{R}^3, m^{-1}(v) dv), \quad \mathcal{Y} = L^1_1(m^{-1}) = L^1(\mathbb{R}^3, \langle v \rangle m^{-1}(v) dv) \quad (2.12)$$

where

$$\langle v \rangle = (1 + |v|^2)^{\frac{1}{2}} \quad \text{and} \quad m(v) = \exp(-a|v|), \quad a > 0, \quad v \in \mathbb{R}^3.$$

According to the above Theorem 2.1,  $F_\alpha \in \mathcal{X}$  for any  $\alpha \in (0, 1]$ . We recall from [1, Proposition 11] that  $\mathcal{Q}_\alpha$  is well defined on  $\mathcal{Y}$ :

**Proposition 2.3.** *There exists  $C > 0$  such that, for any  $\alpha \in (0, 1)$*

$$\|\mathcal{Q}_\alpha(h, g)\|_{\mathcal{X}} + \|\mathcal{Q}_\alpha(g, h)\|_{\mathcal{X}} \leq C \|h\|_{\mathcal{Y}} \|g\|_{\mathcal{Y}} \quad \forall h, g \in \mathcal{Y}.$$

Moreover, one has the following:

**Proposition 2.4** ([16, Proposition 3.2]). *For any  $\alpha, \alpha' \in (0, 1)$  and any  $f \in W^{1,1}_1(m^{-1})$ ,  $g \in L^1_1(m^{-1})$ , it holds*

$$\|\mathcal{Q}_\alpha^+(f, g) - \mathcal{Q}_{\alpha'}^+(f, g)\|_{\mathcal{X}} \leq p(\alpha - \alpha') \|f\|_{W^{1,1}_1(m^{-1})} \|g\|_{\mathcal{Y}}$$

and

$$\|\mathcal{Q}_\alpha^+(g, f) - \mathcal{Q}_{\alpha'}^+(g, f)\|_{\mathcal{X}} \leq p(\alpha - \alpha') \|f\|_{W^{1,1}_1(m^{-1})} \|g\|_{\mathcal{Y}}$$

where  $p(r)$  is an explicit polynomial function with  $\lim_{r \rightarrow 0^+} p(r) = 0$ .

In the elastic limit  $\alpha \rightarrow 1$ , one has the following:

**Theorem 2.5** ([4, Theorem 5.5]). *There exists an explicit function  $\eta_1(\alpha)$  such that  $\lim_{\alpha \rightarrow 1} \eta_1(\alpha) = 0$  and such that*

$$\|F_\alpha - \mathcal{M}\|_{\mathcal{Y}} \leq \eta_1(\alpha) \quad \forall \alpha \in (\alpha_0, 1].$$

**2.4. Spectral properties of the linearised operator for  $\alpha = 1$ .** Define the elastic linearised operator  $\mathcal{L}_1 : \mathcal{D}(\mathcal{L}_1) \subset \mathcal{X} \rightarrow \mathcal{X}$  by

$$\mathcal{L}_1(h) = \mathcal{Q}_1(\mathcal{M}, h) + \mathcal{Q}_1(h, \mathcal{M}) + \mathbf{L}h, \quad \forall h \in \mathcal{D}(\mathcal{L}_1) = \mathcal{Y}.$$

(We recall  $\mathcal{X}$  and  $\mathcal{Y}$  were defined in (2.12).) We introduce also

$$\widehat{\mathcal{X}} = \{f \in \mathcal{X}; \int_{\mathbb{R}^3} f dv = 0\}, \quad \widehat{\mathcal{Y}} = \{f \in \mathcal{Y}; \int_{\mathbb{R}^3} f dv = 0\}.$$

One has the following structure of the spectrum of  $\mathcal{L}_1$ :

**Theorem 2.6** ([4, Theorem 5.3]). *The null space of  $\mathcal{L}_1$  in  $\mathcal{X}$  is given by*

$$\mathcal{N}(\mathcal{L}_1) = \text{span}(\mathcal{M}).$$

Moreover,  $\mathcal{L}_1$  admits a positive spectral gap  $\nu > 0$ . In particular,  $\mathcal{N}(\mathcal{L}_1) \cap \widehat{\mathcal{X}} = \{0\}$  and  $\mathcal{L}_1$  is invertible from  $\widehat{\mathcal{Y}}$  to  $\widehat{\mathcal{X}}$ .

Let us spend a few words on the strategy used to prove the above result since we will use several of the tools involved to study the spectral properties of the linearised semi-group in the next section. The proof of the above result is related to a general strategy introduced in [12] which consists in deducing the spectral properties in  $L^1$  from the much easier spectral analysis in  $L^2$ . The existence of a spectral gap for the linearised collision operator in  $\mathcal{H} = L^2(\mathcal{M}^{-1})$  is relatively easy to obtain through a suitable Poincaré-like inequality and the task is to prove that the linearised collision operator in  $\mathcal{X}$  can be deduced from the one in the Hilbert setting. This is done thanks to a suitable splitting of the linearised operator as

$$\mathcal{L}_1 = \mathcal{A}_1 + \mathcal{B}_1$$

where

- (i)  $\mathcal{A}_1 : \mathcal{X} \rightarrow \mathcal{H}$  is bounded;
- (ii) the operator  $\mathcal{B}_1 : \mathcal{D}(\mathcal{B}_1) \rightarrow \mathcal{X}$  (with  $\mathcal{D}(\mathcal{B}_1) = \mathcal{Y}$ ) is  $\beta$ -dissipative for some positive  $\beta > 0$ , i.e.

$$\int_{\mathbb{R}^3} \text{sign} f(v) \mathcal{B}_1 f(v) m^{-1}(v) dv \leq -\beta \|f\|_{\mathcal{Y}} \quad \forall f \in \mathcal{Y}. \quad (2.13)$$

Under these conditions [12] asserts that the spectrum of  $\mathcal{L}_1$  in  $\mathcal{X}$  will be the same of that in  $\mathcal{H}$ .

To be more precise, the splitting is as follows. Let us introduce the linearised Boltzmann operator

$$\mathcal{T}_1 f = \mathcal{Q}_1(\mathcal{M}, f) + \mathcal{Q}_1(f, \mathcal{M})$$

so that  $\mathcal{L}_1 = \mathcal{T}_1 + \mathbf{L}$ . Clearly,

$$\mathcal{T}_1 f = \mathcal{T}_1^+ f - \sigma_1(v) f(v) - \mathcal{M}(v) \int_{\mathbb{R}^3} f(w) |v-w| dw, \quad \text{while} \quad \mathbf{L} f(v) = \mathbf{L}^+(f) - \Sigma(v) f(v)$$

and the splitting consists in setting, for some  $R > 0$  large enough so that (2.13) holds,

$$\mathcal{A}_1 = \mathcal{A}_1^1 + \mathcal{A}_1^2$$

with

$$\mathcal{A}_1^1 f = \mathcal{T}_1^+(\chi_{B_R} f) + \mathbf{L}^+(\chi_{B_R} f), \quad \mathcal{A}_1^2 f(v) = -\mathcal{M}(v) \int_{\mathbb{R}^3} f(w) |v-w| dw$$

where  $B_R$  is the open ball in  $\mathbb{R}^3$  with radius  $R > 0$  and center 0. Then, simply sets  $\mathcal{B}_1 = \mathcal{L}_1 - \mathcal{A}_1$ . Notice that, in the above inequality (2.13), the constant  $\beta > 0$  can be chosen as

$$\beta = \underline{\Sigma} + \underline{\sigma}_1 + \varepsilon$$

for some arbitrarily small  $\varepsilon > 0$  where

$$\underline{\Sigma} = \inf_{v \in \mathbb{R}^3} \frac{\Sigma(v)}{1+|v|} > 0 \quad \text{and} \quad \underline{\sigma}_1 = \inf_{v \in \mathbb{R}^3} \frac{\sigma_1(v)}{1+|v|} > 0.$$

Notice  $\beta$  does not depend on  $R$ .

### 3. SPECTRAL ANALYSIS OF THE LINEARISED OPERATOR AND ITS ASSOCIATED SEMIGROUP

We recall that for  $\alpha \in (\alpha_0, 1]$ ,  $F_\alpha$  denotes the unique steady state with unit mass, solution to (1.3). In order to study the stability of  $F_\alpha$  for  $\alpha$  close to 1 we will first prove that the following operator has a spectral gap in  $\mathcal{X}$ :

$$\mathcal{L}_\alpha(h) := \mathcal{Q}_\alpha(h, F_\alpha) + \mathcal{Q}_\alpha(F_\alpha, h) + \mathbf{L}(h) \quad (h \in \mathcal{Y}). \quad (3.1)$$

Notice that thanks to Proposition 2.3 the above expression is well defined and belongs to  $\mathcal{X}$  whenever  $h \in \mathcal{Y}$ . It is fundamental here that the domain of  $\mathcal{L}_\alpha$  in  $\mathcal{X}$  does not depend on  $\alpha$ , i.e.

$$\mathcal{D}(\mathcal{L}_\alpha) = \mathcal{D}(\mathcal{L}_1) = \mathcal{Y} \quad \forall \alpha \in (\alpha_0, 1].$$

This is in major contrast with the situations investigated in [16, 17] where the forcing term was a differential operator for which the domain of the associated linearised operator involved Sobolev norms.

Since  $\mathcal{L}_\alpha$  is the linearisation of the nonlinear operator  $\mathcal{Q}_\alpha(f, f) + \mathbf{L}(f)$  near its steady state  $F_\alpha$ , a study of its spectrum will allow us to perform a perturbative study of the evolution equation (2.8). We deduce from the results of Section 2.3 the following technical result stating that  $\mathcal{L}_\alpha$  is close to  $\mathcal{L}_1$  for  $\alpha$  close to 1. It will play a crucial role in our analysis:

**Proposition 3.1.** *There exists an explicit function  $\varpi: (\alpha_0, 1] \rightarrow \mathbb{R}^+$  such that  $\lim_{\alpha \rightarrow 1^+} \varpi(\alpha) = 0$  and*

$$\|\mathcal{L}_\alpha(h) - \mathcal{L}_1(h)\|_{\mathcal{X}} \leq \varpi(\alpha) \|h\|_{\mathcal{Y}} \quad \forall h \in \mathcal{Y}.$$

*Proof.* A fundamental observation is that the domain of  $\mathcal{L}_\alpha$  is actually independent of  $\alpha$ , i.e.  $\mathcal{D}(\mathcal{L}_\alpha) = \mathcal{Y}$  for any  $\alpha \in (\alpha_0, 1]$ . Let  $h \in \mathcal{Y}$  be fixed. We have

$$\begin{aligned} \|\mathcal{L}_\alpha(h) - \mathcal{L}_1(h)\|_{\mathcal{X}} &= \|\mathcal{Q}_\alpha(F_\alpha, h) + \mathcal{Q}_\alpha(h, F_\alpha) - \mathcal{Q}_1(\mathcal{M}, h) - \mathcal{Q}_1(h, \mathcal{M})\|_{\mathcal{X}} \\ &\leq \|\mathcal{Q}_\alpha(F_\alpha - \mathcal{M}, h) + \mathcal{Q}_\alpha(h, F_\alpha - \mathcal{M})\|_{\mathcal{X}} \\ &\quad + \|\mathcal{Q}_\alpha(\mathcal{M}, h) - \mathcal{Q}_1(\mathcal{M}, h)\|_{\mathcal{X}} + \|\mathcal{Q}_\alpha(h, \mathcal{M}) - \mathcal{Q}_1(h, \mathcal{M})\|_{\mathcal{X}}. \end{aligned}$$

Thus, using Proposition 2.3 and Theorem 2.5, one sees that there exists some constant  $C > 0$  and some explicit function  $\eta_0(\alpha)$  with  $\lim_{\alpha \rightarrow 1} \eta_0(\alpha) = 0$  such that

$$\|\mathcal{Q}_\alpha(F_\alpha - \mathcal{M}, h) + \mathcal{Q}_\alpha(h, F_\alpha - \mathcal{M})\|_{\mathcal{X}} \leq C \|F_\alpha - \mathcal{M}\|_{\mathcal{Y}} \|h\|_{\mathcal{Y}} \leq \eta_0(\alpha) \|h\|_{\mathcal{Y}}.$$

In the same way, according to Proposition 2.4,

$$\|\mathcal{Q}_\alpha(\mathcal{M}, h) - \mathcal{Q}_1(\mathcal{M}, h)\|_{\mathcal{X}} + \|\mathcal{Q}_\alpha(h, \mathcal{M}) - \mathcal{Q}_1(h, \mathcal{M})\|_{\mathcal{X}} \leq \eta_1(\alpha) \|h\|_{\mathcal{Y}}$$

some explicit function  $\eta_1(\alpha)$  with  $\lim_{\alpha \rightarrow 1} \eta_1(\alpha) = 0$ . These two estimates give the result with  $\varpi(\cdot) = \eta_0(\cdot) + \eta_1(\cdot)$ .  $\square$

**3.1. Spectral gap of the linearised operator.** We investigate here the spectral properties of  $\mathcal{L}_\alpha$  in  $\mathcal{X}$ . We begin by studying the kernel of  $\mathcal{L}_\alpha$ :

**Proposition 3.2.** *There exists some explicit  $\alpha_1 \in (\alpha_0, 1]$  such that, given  $\alpha \in (\alpha_1, 1]$ , 0 is a simple and isolated eigenvalue of  $\mathcal{L}_\alpha$  and there exists  $G_\alpha \in \mathcal{Y}$  with unit mass and such that*

$$\mathcal{N}(\mathcal{L}_\alpha) = \text{Span}(G_\alpha) \quad \forall \alpha \in (\alpha_1, 1].$$

We denote then by  $\mathbb{P}_\alpha$  the spectral projection associated to the zero eigenvalue of  $\mathcal{L}_\alpha$ . Then, for any  $f \in \mathcal{X}$ , one has  $\mathbb{P}_\alpha f = \varrho_f G_\alpha$  with  $\varrho_f := \int_{\mathbb{R}^3} f(v) dv$ . In particular,  $\text{Range}(\mathbb{I} - \mathbb{P}_\alpha) = \hat{\mathcal{X}}$  for any  $\alpha \in (\alpha_1, 1]$ .

*Proof.* For any  $\alpha \in (\alpha_0, 1]$ , set for simplicity  $T_\alpha = \mathcal{L}_1 - \mathcal{L}_\alpha$  with domain  $\mathcal{D}(T_\alpha) = \mathcal{Y}$ . From Proposition 3.1,

$$\|T_\alpha h\|_{\mathcal{X}} \leq \varpi(\alpha) \|h\|_{\mathcal{Y}} \quad \forall h \in \mathcal{D}(\mathcal{L}_1) = \mathcal{Y}.$$

Since  $\|\cdot\|_{\mathcal{Y}}$  is equivalent to the graph norm of  $\mathcal{D}(\mathcal{L}_1)$ , there exists  $c > 0$  such that

$$\|h\|_{\mathcal{Y}} \leq c(\|h\|_{\mathcal{X}} + \|\mathcal{L}_1 h\|_{\mathcal{X}}) \quad \forall h \in \mathcal{Y}$$

from which the above inequality reads

$$\|T_\alpha h\|_{\mathcal{X}} \leq a\|h\|_{\mathcal{X}} + b\|\mathcal{L}_1 h\|_{\mathcal{X}} \quad \forall h \in \mathcal{D}(\mathcal{L}_1)$$

with  $b = c\varpi(\alpha)$ . Since  $\lim_{\alpha \rightarrow 1+} \varpi(\alpha) = 0$ , this makes  $T_\alpha$  a  $\mathcal{L}_1$ -bounded operator with relative bound  $b < 1$  for any  $\alpha \in (\alpha'_0, 1]$  for some explicit  $\alpha'_0 \in (\alpha_0, 1]$ . In particular, according to [13, Theorem 2.14, p. 203] (cf. Theorem A.2), the gap  $\hat{\delta}(\mathcal{L}_\alpha, \mathcal{L}_1)$  between  $\mathcal{L}_\alpha = \mathcal{L}_1 + T_\alpha$  and  $\mathcal{L}_1$  (as defined in [13, IV.2.4, p. 201]; see Appendix A) is less than  $\frac{\sqrt{2b^2}}{1-b} = \frac{\sqrt{2c\varpi(\alpha)}}{1-c\varpi(\alpha)}$ . Now, recall that the spectrum of  $\mathcal{L}_1$  splits as

$$\mathfrak{S}(\mathcal{L}_1) = \{0\} \cup \mathfrak{S}'(\mathcal{L}_1)$$

where  $\mathfrak{S}'(\mathcal{L}_1) \subset \{z \in \mathbb{C}; \text{Re } z \leq -\nu\}$ . Denoting by  $\mathbb{P}_1$  the spectral projection associated to the 0 eigenvalue, one gets that  $\mathcal{X} = \mathcal{X}'' \oplus \mathcal{X}'$  with  $\mathcal{X}'' = \text{Range}(\mathbb{P}_1)$  and  $\mathcal{X}' = \text{Range}(\mathbb{I} - \mathbb{P}_1)$  with moreover  $\mathfrak{S}(\mathcal{L}_1|_{\mathcal{X}''}) = \{0\}$  and  $\mathfrak{S}(\mathcal{L}_1|_{\mathcal{X}'}) = \mathfrak{S}'(\mathcal{L}_1)$ . In particular, the two above parts of  $\mathfrak{S}(\mathcal{L}_1)$  are separated by the closed curve  $\gamma_r = \{z \in \mathbb{C}; |z| = r\}$ , for any  $r \in (0, \nu)$ . Then, according to [13, Theorem 3.16, p. 212 & IV.3.5] (see Theorem A.3), there exists  $\delta > 0$  such that the same separation of the spectrum and decomposition of  $\mathcal{X}$  hold for any operator  $S \in \mathcal{C}(\mathcal{X})$  for which the gap  $\hat{\delta}(S, \mathcal{L}_1) < \delta$ . Choosing now  $\alpha''_0 \in (\alpha'_0, 1]$  such that  $\hat{\delta}(\mathcal{L}_\alpha, \mathcal{L}_1) < \delta$  as soon as  $\alpha \in (\alpha''_0, 1]$ , one gets therefore that the spectrum of  $\mathfrak{S}(\mathcal{L}_\alpha)$  can be separated by  $\gamma_r$ , i.e. it splits as

$$\mathfrak{S}(\mathcal{L}_\alpha) = \mathfrak{S}''(\mathcal{L}_\alpha) \cup \mathfrak{S}'(\mathcal{L}_\alpha) \quad \forall \alpha \in (\alpha''_0, 1]$$

where  $\mathfrak{S}''(\mathcal{L}_\alpha) \subset \{z \in \mathbb{C}; |z| < r\}$  while  $\mathfrak{S}'(\mathcal{L}_\alpha) \subset \{z \in \mathbb{C}; |z| > r\}$ . Moreover, the space  $\mathcal{X}$  splits as  $\mathcal{X} = \mathcal{X}''_\alpha \oplus \mathcal{X}'_\alpha$  with  $\mathfrak{S}(\mathcal{L}_\alpha|_{\mathcal{X}''_\alpha}) = \mathfrak{S}''(\mathcal{L}_\alpha)$  and  $\mathfrak{S}(\mathcal{L}_\alpha|_{\mathcal{X}'_\alpha}) = \mathfrak{S}'(\mathcal{L}_\alpha)$ . Moreover, still using Theorem A.3,  $\dim(\mathcal{X}''_\alpha) = \dim(\mathcal{X}''_\alpha) = 1$ . This shows that actually

$$\mathfrak{S}''(\mathcal{L}_\alpha) = \{\mu_\alpha\}$$

where  $\mu_\alpha$  is a *simple eigenvalue* of  $\mathcal{L}_\alpha$  with  $|\mu_\alpha| < r$  for any  $\alpha \in (\alpha_0'', 1]$ . Let us show that actually  $\mu_\alpha = 0$  (at least for sufficiently large  $\alpha$ )<sup>1</sup>. Define  $\mathbb{P}_\alpha$  as the spectral projection operator associated to  $\mu_\alpha$ , i.e.

$$\mathbb{P}_\alpha = \frac{1}{2\pi i} \oint_{\gamma_r} R(\xi, \mathcal{L}_\alpha) d\xi \quad \forall \alpha \in (\alpha_1, 1].$$

According to A.3, one also has  $\lim_{\alpha \rightarrow 1} \|\mathbb{P}_\alpha - \mathbb{P}_1\|_{\mathcal{B}(\mathcal{X})} = 0$  with an explicit rate, from which there exists some explicit  $\alpha_1 \in (\alpha_0'', 1]$  such that

$$\|\mathbb{P}_\alpha f - \mathbb{P}_1 f\|_{\mathcal{X}} < 1 \quad \text{for all } \alpha \in (\alpha_1, 1], f \in \mathcal{X}. \quad (3.2)$$

Let us prove that  $\mu_\alpha = 0$  for any  $\alpha \in (\alpha_1, 1]$ . Let us argue by contradiction and assume there exists  $\alpha \in (\alpha_1, 1]$  for which  $\mu_\alpha \neq 0$ . Let  $\phi_\alpha$  be some normalized eigenfunction of  $\mathcal{L}_\alpha$  associated to  $\mu_\alpha$ , i.e.  $\phi_\alpha \in \mathcal{Y} \setminus \{0\}$  satisfies  $\mathcal{L}_\alpha \phi_\alpha = \mu_\alpha \phi_\alpha$ . Integrating over  $\mathbb{R}^3$  we get that

$$\int_{\mathbb{R}^3} \phi_\alpha(v) dv = 0.$$

For any  $f \in \mathcal{X}$ , there exists  $\beta = \beta(\alpha, f)$  such that  $\mathbb{P}_\alpha f = \beta \phi_\alpha$  while  $\mathbb{P}_1 f = \varrho_f \mathcal{M}$ . In particular, one sees that

$$\int_{\mathbb{R}^3} \mathbb{P}_\alpha f dv = 0 \quad \text{while} \quad \int_{\mathbb{R}^3} \mathbb{P}_1 f dv = \varrho_f \quad \forall f \in \mathcal{X}.$$

This clearly contradicts (3.2). Therefore, for any  $\alpha \in (\alpha_1, 1]$ ,  $\mu_\alpha = 0$  and the above reasoning shows that any associated eigenfunction  $\phi_\alpha$  is such that

$$\int_{\mathbb{R}^3} \phi_\alpha(v) dv \neq 0.$$

Let then  $G_\alpha$  be the unique eigenfunction of  $\mathcal{L}_\alpha$  associated to the 0 eigenvalue with  $\int_{\mathbb{R}^3} G_\alpha(v) dv = 1$ . From [13, Eq. (6.34), p. 180], one has  $\text{Range}(\mathbb{I} - \mathbb{P}_\alpha) \subset \text{Range}(\mathcal{L}_\alpha)$  for any  $\alpha \in (\alpha_2, 1]$ . Since  $\int_{\mathbb{R}^3} \mathcal{L}_\alpha f dv = 0$  for any  $f \in \mathcal{D}(\mathcal{L}_\alpha)$  we get that  $\text{Range}(\mathbb{I} - \mathbb{P}_\alpha) \subset \hat{\mathcal{X}}$  for any  $\alpha_2 < \alpha \leq 1$ . Thus, given  $f \in \mathcal{X}$ , since  $f = \mathbb{P}_\alpha f + (\mathbb{I} - \mathbb{P}_\alpha)f$ , it holds

$$\varrho_f := \int_{\mathbb{R}^3} f dv = \int_{\mathbb{R}^3} \mathbb{P}_\alpha f dv.$$

Since moreover  $\mathbb{P}_\alpha f = \beta_f G_\alpha$  for some  $\beta_f \in \mathbb{R}$  and  $G_\alpha$  is normalised, we get that  $\mathbb{P}_\alpha f = \varrho_f G_\alpha$ . In particular, if  $f \in \hat{\mathcal{X}}$  then  $\mathbb{P}_\alpha f = 0$  and  $f \in \text{Range}(\mathbb{I} - \mathbb{P}_\alpha)$  which achieves the proof of the result.  $\square$

From now on,  $\nu > 0$  will denote the size of the spectral gap of  $\mathcal{L}_1$  (see Theorem 2.6). Our main result in this subsection is the following:

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<sup>1</sup>Notice that this cannot be deduced directly from the fact that  $r > 0$  can be chosen arbitrarily small since the range of parameters  $(\alpha_0'', 1]$  for which the above splitting holds actually depends on  $r$  through the parameter  $\delta$  in A.3.

**Theorem 3.3.** Take  $0 < \nu_* < \nu$ . There is  $\alpha_0 < \alpha_* < 1$ , with  $\alpha_*$  depending on  $\nu_*$ , such that for all  $\alpha_* < \alpha \leq 1$ , the linear operator  $\mathcal{L}_\alpha$  has a spectral gap of size  $\nu_*$ . More precisely, the spectrum of  $\mathcal{L}_\alpha$  splits as  $\mathfrak{S}(\mathcal{L}_\alpha) = \{0\} \cup \mathfrak{S}(\mathcal{L}_\alpha|_{(\mathbb{I}-\mathbb{P}_\alpha)\mathcal{X}})$  with

$$\mathfrak{S}(\mathcal{L}_\alpha|_{\hat{\mathcal{X}}}) \subset \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \leq -\nu_*\} \quad \forall \alpha \in (\alpha_*, 1] \quad (3.3)$$

where we recall that  $\mathbb{P}_\alpha$  denotes the spectral projection associated to the zero eigenvalue of  $\mathcal{L}_\alpha$  and  $\mathcal{L}_\alpha|_{\hat{\mathcal{X}}}$  denotes the part of  $\mathcal{L}_\alpha$  on  $\hat{\mathcal{X}} = (\mathbb{I} - \mathbb{P}_\alpha)\mathcal{X}$ .

To prove this we will use the following result asserting that if an operator has a spectral gap, and another operator is close to it in a certain sense, then it must also have a spectral gap of a comparable size:

**Lemma 3.4.** Let  $X$  be a Banach space and let  $(L_0, \mathcal{D}(L_0))$  be the generator of a  $C_0$ -semigroup. For any  $\varepsilon \in (0, 1)$  let  $(L_\varepsilon, \mathcal{D}(L_\varepsilon))$  be a given closed unbounded operator with  $\mathcal{D}(L_0) \subset \mathcal{D}(L_\varepsilon)$  and

$$\lim_{\varepsilon \rightarrow 0} \|(L_\varepsilon - L_0)R(\lambda, L_0)\|_{\mathcal{B}(X)} = 0 \quad \forall \lambda \in \mathbb{C} \quad \text{with } \operatorname{Re} \lambda > s(L_0). \quad (3.4)$$

Then,

$$\limsup_{\varepsilon \rightarrow 0} s(L_\varepsilon) \leq s(L_0)$$

with

$$\lim_{\varepsilon \rightarrow 0} \|R(\lambda, L_\varepsilon) - R(\lambda, L_0)\|_{\mathcal{B}(X)} = 0 \quad \forall \lambda \in \mathbb{C} \quad \text{with } \operatorname{Re} \lambda > s(L_0).$$

*Proof.* Let  $\lambda \in \mathbb{C}$  be given with  $\operatorname{Re} \lambda > s(L_0)$  and let  $\varepsilon_0 > 0$  be such that

$$\|(L_0 - L_\varepsilon)R(\lambda, L_0)\|_{\mathcal{B}(X)} < 1 \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Setting

$$\mathcal{J}_\varepsilon = I - (L_\varepsilon - L_0)R(\lambda, L_0) =: I - Z_\varepsilon$$

one gets that  $\mathcal{J}_\varepsilon$  is invertible for any  $\varepsilon \in (0, \varepsilon_0)$  with  $\mathcal{J}_\varepsilon^{-1} = \sum_{n=0}^{\infty} Z_\varepsilon^n$ . Since moreover

$$\mathcal{J}_\varepsilon = (\lambda - L_\varepsilon)R(\lambda, L_0)$$

one gets that  $\lambda - L_\varepsilon$  is invertible for any  $\varepsilon \in (0, \varepsilon_0)$  with

$$R(\lambda, L_\varepsilon) = R(\lambda, L_0)\mathcal{J}_\varepsilon^{-1} \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Therefore,  $\operatorname{Re} \lambda \geq s(L_\varepsilon)$  for any  $\varepsilon \in (0, \varepsilon_0)$  which proves the first part of the result. For the second part, one has simply, for a given  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda > s(L_0)$ ,

$$\|R(\lambda, L_\varepsilon) - R(\lambda, L_0)\|_{\mathcal{B}(X)} = \|R(\lambda, L_0)(\mathcal{J}_\varepsilon^{-1} - I)\|_{\mathcal{B}(X)}$$

and it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \|\mathcal{J}_\varepsilon^{-1} - I\|_{\mathcal{B}(X)} = 0.$$

Since  $\mathcal{J}_\varepsilon^{-1} = \sum_{n=0}^{\infty} Z_\varepsilon^n$ , we get

$$\|\mathcal{J}_\varepsilon^{-1} - I\|_{\mathcal{B}(X)} \leq \sum_{n=1}^{\infty} \|Z_\varepsilon^n\|_{\mathcal{B}(X)}$$

and clearly, since  $\lim_{\varepsilon \rightarrow 0} \|Z_\varepsilon\|_{\mathcal{B}(X)} = 0$  we get the conclusion.  $\square$

**Remark 3.5.** We notice that the above result can also be seen as a way of stating general abstract results for relatively bounded operators (see [13, Theorem 3.17, p. 214]).

We are now ready to prove Theorem 3.3:

*Proof of Theorem 3.3.* We will apply Lemma 3.4 to the restriction of  $\mathcal{L}_\alpha$  to  $\widehat{\mathcal{X}}$  for  $\alpha$  close to 1. (We recall the reader that the spaces  $\widehat{\mathcal{X}}$  and  $\widehat{\mathcal{Y}}$  were defined in section 2.4.) Notice that, since

$$\int_{\mathbb{R}^3} \mathcal{L}_\alpha f(v) dv = 0 \quad \forall f \in \mathcal{D}(\mathcal{L}_\alpha) \quad \alpha \in (0, 1]$$

one can define the restriction  $\widehat{\mathcal{L}}_\alpha: \mathcal{D}(\widehat{\mathcal{L}}_\alpha) \subset \widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}$  by  $\mathcal{D}(\widehat{\mathcal{L}}_\alpha) = \mathcal{D}(\mathcal{L}_\alpha) \cap \widehat{\mathcal{X}} = \widehat{\mathcal{Y}}$  and  $\widehat{\mathcal{L}}_\alpha f = \mathcal{L}_\alpha f$  for any  $f \in \widehat{\mathcal{Y}}$  for any  $\alpha \in (0, 1]$ . According to Theorem 2.6,  $s(\widehat{\mathcal{L}}_1) = -\nu < 0$ .

Estimate (3.4) in Lemma 3.4 for  $\widehat{\mathcal{L}}_1$  and  $\widehat{\mathcal{L}}_\alpha$  is exactly Proposition 3.1 since

$$\|\widehat{\mathcal{L}}_\alpha(h) - \widehat{\mathcal{L}}_1(h)\|_{\widehat{\mathcal{X}}} = \|\mathcal{L}_\alpha(h) - \mathcal{L}_1(h)\|_{\mathcal{X}} \quad \forall h \in \widehat{\mathcal{Y}}.$$

Since  $R(\lambda, \widehat{\mathcal{L}}_1): \widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{Y}}$ , the hypotheses of Lemma 3.4 are satisfied and therefore  $s(\widehat{\mathcal{L}}_\alpha) \leq \nu_* < \nu = s(\widehat{\mathcal{L}}_1)$  for any  $\alpha$  close enough to 1. Now, since  $\widehat{\mathcal{X}} = \text{Range}(\mathbb{I} - \mathbb{P}_\alpha)$  for any  $\alpha \in (\alpha_1, 1]$ , one has  $\widehat{\mathcal{L}}_\alpha = \mathcal{L}_\alpha|_{(\mathbb{I} - \mathbb{P}_\alpha)\mathcal{X}}$  for any  $\alpha \in (\alpha_1, 1]$ . This finishes the proof.  $\square$

**3.2. Decay of the associated semigroup.** Now, one should translate the above spectral gap of the operator  $\mathcal{L}_\alpha$  into a decay of the associated semigroup. To do so, we use a stable splitting of the generator  $\mathcal{L}_\alpha$  into a dissipative part and a regularising part. This strategy is inspired in the recent results [12, 18], but we give a proof adapted to our situation, exploiting a well known stability property of the essential spectrum under weakly compact perturbations. Our splitting is in the spirit of the one described in Section 2.4. Namely, set

$$\mathcal{T}_\alpha f = \mathcal{Q}_\alpha(f, F_\alpha) + \mathcal{Q}_\alpha(F_\alpha, f), \quad f \in \mathcal{Y}$$

so that  $\mathcal{L}_\alpha = \mathcal{T}_\alpha + \mathbf{L}$ . The positive part of this operator is

$$\mathcal{T}_\alpha^+ f = \mathcal{Q}_\alpha^+(f, F_\alpha) + \mathcal{Q}_\alpha^+(F_\alpha, f)$$

and  $\mathcal{T}_\alpha$  is written as

$$\mathcal{T}_\alpha f(v) = \mathcal{T}_\alpha^+ f(v) - F_\alpha(v) \int_{\mathbb{R}^3} f(w) |v - w| dw - \sigma_\alpha(v) f(v), \quad v \in \mathbb{R}^3, f \in \mathcal{Y},$$

where

$$\sigma_\alpha(v) = \int_{\mathbb{R}^3} F_\alpha(w) |v - w| dw \geq \underline{\sigma}_\alpha (1 + |v|) \quad \forall v \in \mathbb{R}^3.$$

Inspired by the splitting in Section 2.4, let us pick  $R > 0$  large enough so that (2.13) holds true and define, for any  $\alpha$ ,

$$\mathcal{B}_\alpha f(v) = \mathcal{T}_\alpha^+(\chi_{B_R^c} f)(v) + \mathbf{L}^+(\chi_{B_R^c} f)(v) - (\Sigma(v) + \sigma_\alpha(v)) f(v) \quad (3.5)$$

for  $f \in \mathcal{Y}$  and  $v \in \mathbb{R}^3$ . Let us first see that  $\mathcal{B}_\alpha$  thus defined is dissipative:

**Lemma 3.6.** *Let  $R > 0$  and  $\beta > 0$  be given as in (2.13). For any  $0 < \beta_* < \beta$ , there exists  $\alpha^\dagger = \alpha^\dagger(\beta_*) \in (\alpha_0, 1)$  such that*

$$\int_{\mathbb{R}^3} \text{sign} f(v) \mathcal{B}_\alpha f(v) m^{-1}(v) dv \leq -\beta_* \|f\|_{\mathcal{Y}} \quad \forall f \in \mathcal{Y}, \quad \forall \alpha \in (\alpha^\dagger, 1). \quad (3.6)$$

*Proof.* The proof is a direct consequence of (2.13) together with the fact that  $\mathcal{T}_\alpha^+$  converges strongly to  $\mathcal{T}_1^+$ . More precisely, let us fix  $f \in \mathcal{Y}$  and compute first  $\|\mathcal{T}_\alpha^+ f_R - \mathcal{T}_1^+ f_R\|_{\mathcal{X}}$ . One checks easily that

$$\begin{aligned} \|\mathcal{T}_\alpha^+ f_R - \mathcal{T}_1^+ f_R\|_{\mathcal{X}} &\leq \|\mathcal{Q}_\alpha^+(F_\alpha - \mathcal{M}, f_R) + \mathcal{Q}_\alpha^+(f_R, F_\alpha - \mathcal{M})\|_{\mathcal{X}} \\ &\quad + \|\mathcal{Q}_\alpha^+(\mathcal{M}, f_R) - \mathcal{Q}_1^+(\mathcal{M}, f_R)\|_{\mathcal{X}} + \|\mathcal{Q}_\alpha^+(f_R, \mathcal{M}) - \mathcal{Q}_1^+(f_R, \mathcal{M})\|_{\mathcal{X}} \\ &\leq C \|F_\alpha - \mathcal{M}\|_{\mathcal{Y}} \|f_R\|_{\mathcal{Y}} + 2p(1 - \alpha) \|\mathcal{M}\|_{W_1^{1,1}(m^{-1})} \|f_R\|_{\mathcal{Y}} \end{aligned}$$

where we used both Proposition 2.3 and 2.4. Consequently, there exists a nonnegative function  $\delta_1(\alpha)$  with  $\lim_{\alpha \rightarrow 1} \delta_1(\alpha) = 0$  such that

$$\|\mathcal{T}_\alpha^+ f_R - \mathcal{T}_1^+ f_R\|_{\mathcal{X}} \leq \delta_1(\alpha) \|f\|_{\mathcal{Y}} \quad \forall \alpha \in (\alpha_0, 1) \quad \forall R > 0. \quad (3.7)$$

Set now

$$\mathcal{F}_\alpha(f) = \int_{\mathbb{R}^3} \text{sign} f(v) \mathcal{B}_\alpha f(v) m^{-1}(v) dv \quad \forall \alpha \in (\alpha_0, 1].$$

Using the fact that

$$\mathcal{B}_\alpha f(v) - \mathcal{B}_1 f(v) = \mathcal{T}_\alpha^+ f_R(v) - \mathcal{T}_1^+ f_R(v) - (\sigma_\alpha(v) - \sigma_1(v)) f(v)$$

we get readily that

$$\begin{aligned} \mathcal{F}_\alpha(f) &\leq \mathcal{F}_1(f) + \|\mathcal{T}_\alpha^+ f_R - \mathcal{T}_1^+ f_R\|_{\mathcal{X}} \\ &\quad - \int_{\mathbb{R}^3} (\sigma_\alpha(v) - \sigma_1(v)) |f(v)| m^{-1}(v) dv \\ &\leq \mathcal{F}_1(f) + \delta_1(\alpha) \|f\|_{\mathcal{Y}} + \int_{\mathbb{R}^3} |\sigma_\alpha(v) - \sigma_1(v)| |f(v)| m^{-1}(v) dv \end{aligned}$$

where we used (3.7). Finally, since  $|v - w| \leq \langle v \rangle \langle w \rangle \quad \forall v, w \in \mathbb{R}^3$ , we have

$$|\sigma_\alpha(v) - \sigma_1(v)| \leq \int_{\mathbb{R}^3} |v - w| |F_\alpha(w) - \mathcal{M}(w)| dw \leq \langle v \rangle \|F_\alpha - \mathcal{M}\|_{L_1^1(\mathbb{R}^3)} \leq \langle v \rangle \|F_\alpha - \mathcal{M}\|_{\mathcal{Y}}$$

and we deduce from Theorem 2.5 that

$$\int_{\mathbb{R}^3} |\sigma_\alpha(v) - \sigma_1(v)| |f(v)| m^{-1}(v) dv \leq \eta_1(\alpha) \int_{\mathbb{R}^3} \langle v \rangle |f(v)| m^{-1}(v) dv = \eta_1(\alpha) \|f\|_{\mathcal{Y}}$$

with  $\lim_{\alpha \rightarrow 1} \eta_1(\alpha) = 0$ . To summarize, there exists a function  $\delta(\cdot)$  with  $\lim_{\alpha \rightarrow 1} \delta(\alpha) = 0$  such that

$$\mathcal{F}_\alpha(f) \leq \mathcal{F}_1(f) + \delta(\alpha) \|f\|_{\mathcal{Y}} \quad \forall f \in \mathcal{Y}$$

which, from (2.13), becomes

$$\mathcal{I}_\alpha(f) \leq (\delta(\alpha) - \beta) \|f\|_{\mathcal{Y}} \quad \forall f \in \mathcal{Y}.$$



This gives the first part of result since  $\lim_{\alpha \rightarrow 1} \delta(\alpha) = 0$ .  $\square$

We set now  $\mathcal{A}_\alpha = \mathcal{L}_\alpha - \mathcal{B}_\alpha$ ,  $\alpha \in (\alpha^\dagger, 1)$ , or in other words

$$\mathcal{A}_\alpha f(v) = \mathcal{T}_\alpha^+(\chi_{B_R} f)(v) + \mathbf{L}^+(\chi_{B_R} f)(v) - F_\alpha(v) \int_{\mathbb{R}^3} f(w) |v - w| dw$$

for  $v \in \mathbb{R}^3$  and  $f \in \mathcal{X}$ . Then we have then the following:

**Proposition 3.7.** *For any  $a_\star \in (0, a)$  and for any  $\alpha \in (\alpha^\dagger, 1)$ , one has*

- i)  $\mathcal{A}_\alpha \in \mathcal{B}(X)$ ;
- ii)  $\mathcal{B}_\alpha : \mathcal{D}(\mathcal{B}_\alpha) \subset \mathcal{X} \rightarrow \mathcal{X}$  with domain  $\mathcal{D}(\mathcal{B}_\alpha) = \mathcal{Y}$  is the generator of a  $C_0$ -semigroup  $(\mathcal{U}_\alpha(t))_{t \geq 0}$  of  $\mathcal{X}$  with

$$\|\mathcal{U}_\alpha(t)f\|_{\mathcal{X}} \leq \exp(-\beta_\star t) \|f\|_{\mathcal{X}} \quad \forall t \geq 0, \quad \forall f \in \mathcal{X}. \quad (3.8)$$

*Proof.* It is clear from Proposition 2.3 that  $\mathcal{A}_\alpha$  is a bounded operator in  $\mathcal{X}$  since

$$\begin{aligned} \|\mathcal{A}_\alpha f\|_{\mathcal{X}} &\leq C \|F_\alpha\|_{\mathcal{Y}} \|\chi_{B_R} f\|_{\mathcal{Y}} + \|F_\alpha\|_{\mathcal{Y}} \|f\|_{L^1_1(\mathbb{R}^3)} \\ &\leq C_R \|F_\alpha\|_{\mathcal{Y}} \|f\|_{\mathcal{X}} + \|F_\alpha\|_{\mathcal{Y}} \|f\|_{\mathcal{X}} \quad \forall f \in \mathcal{X} \end{aligned}$$

for some positive constant  $C_R$  depending on  $R$ .

Since  $\mathcal{L}_\alpha$  (with domain  $\mathcal{Y}$ ) is the generator of a  $C_0$ -semigroup in  $\mathcal{X}$  according to Theorem B.3 and  $\mathcal{A}_\alpha$  is bounded, it follows from the classical bounded perturbation theorem that  $\mathcal{B}_\alpha$  (with domain  $\mathcal{Y}$ ) is also the generator of a  $C_0$ -semigroup  $(\mathcal{U}_\alpha(t))_{t \geq 0}$  in  $\mathcal{X}$ . Since  $\mathcal{B}_\alpha + \beta_\star$  is dissipative according to (3.6), (3.8) holds according to the Lumer-Phillips theorem.  $\square$

Actually, it is easy to check that  $\mathcal{A}_\alpha$  has better regularising properties:

**Lemma 3.8.** *For any  $\alpha \in (\alpha_0, 1)$ ,  $\mathcal{A}_\alpha \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ . Moreover, there exists some Maxwellian distribution  $M$  such that, for any  $\alpha \in (\alpha_0, 1)$ ,*

$$\mathcal{A}_\alpha \in \mathcal{B}(\mathcal{X}, \mathcal{H}) \quad \text{where } \mathcal{H} = L^2(M^{-1/2}).$$

*In particular,  $\mathcal{A}_\alpha$  is a weakly compact operator in  $\mathcal{X}$ .*

*Proof.* Recall that there exist two Maxwellian distributions  $\underline{M}$  and  $\overline{M}$  (independent of  $\alpha$ ) such that

$$\underline{M}(v) \leq F_\alpha(v) \leq \overline{M}(v) \quad \forall v \in \mathbb{R}^3, \quad \forall \alpha \in (\alpha_0, 1).$$

In particular, there exists some Maxwellian distribution  $M$  such that

$$\sup_{\alpha \in (\alpha_0, 1)} \|F_\alpha\|_{L^2_2(M^{-1})} = \sup_{\alpha \in (\alpha_0, 1)} \left( \int_{\mathbb{R}^3} M^{-1}(v) \langle v \rangle^2 |f(v)|^2 dv \right)^{1/2} = C_M < \infty.$$

Then, because  $|v - w| \leq \langle v \rangle \langle w \rangle$  for any  $v, w \in \mathbb{R}^3$ , one has first that, for all  $f \in \mathcal{X}$ ,

$$\begin{aligned} \int_{\mathbb{R}^3} \left| F_\alpha(v) \int_{\mathbb{R}^3} f(w) |v - w| dw \right|^2 M^{-1}(v) dv \\ \leq \int_{\mathbb{R}^3} |F_\alpha(v)|^2 \langle v \rangle^2 M^1(v) dv \left( \int_{\mathbb{R}^3} |f(w)| \langle w \rangle dw \right)^2 \\ \leq \|F_\alpha\|_{L^2_2(M^{-1/2})}^2 \|f\|_{L^1_1}^2 \leq \|F_\alpha\|_{L^2_2(M^{-1})}^2 \|f\|_{\mathcal{X}}^2. \end{aligned}$$

Moreover, according to [1, Proposition 11], there exists  $C > 0$  such that

$$\begin{aligned} \|\mathcal{Q}^+(g, h)\|_{L^2(M^{-1})} &\leq C \|g M^{-1/2}\|_{L^1(\mathbb{R}^3)} \|h M^{-1/2}\|_{L^2_1(\mathbb{R}^3)} \quad \text{and} \\ \|\mathcal{Q}^+(h, g)\|_{L^2(M^{-1})} &\leq C \|g M^{-1/2}\|_{L^1_1(\mathbb{R}^3)} \|h M^{-1/2}\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Using this with  $g = f\chi_{B_R}$  and  $h = F_\alpha$  we get that there exists  $C > 0$  (independent of  $\alpha$ ) such that

$$\|\mathcal{L}_\alpha^+(f\chi_{B_R})\|_{L^2(M^{-1})} \leq C \left( \|f\chi_{B_R}\|_{L^1_1(M^{-1/2})} + \|f\chi_{B_R}\|_{L^1(M^{-1/2})} \right).$$

In particular, there exists  $C = C_R > 0$  such that

$$\|\mathcal{L}_\alpha^+(f\chi_{B_R})\|_{L^2(M^{-1})} \leq C_R \|f\|_{\mathcal{X}}.$$

In the same way,

$$\|\mathbf{L}^+(f\chi_{B_R})\|_{L^2(M^{-1})} \leq C_R \|f\|_{\mathcal{X}} \quad \forall f \in \mathcal{X}.$$

This proves that  $\mathcal{A}_\alpha \in \mathcal{B}(\mathcal{X}, \mathcal{H})$ . Due to the Dunford-Pettis theorem the embedding  $\mathcal{H} \hookrightarrow \mathcal{X}$  is weakly compact, which proves the second part of the Lemma.  $\square$

We this in hands, one has the following result about the decay of the semigroup  $(\mathcal{S}_\alpha(t))_{t \geq 0}$  in  $\mathcal{X}$  generated by  $\mathcal{L}_\alpha$ . Remember that  $\nu$  is the spectral gap of the elastic linearised operator  $\mathcal{L}_1$ .

**Theorem 3.9.** *Take  $0 < \nu_* < \nu$  and  $0 < \beta_* < a$  (where  $\beta > 0$  is such that (2.13) holds). With the notations of Theorem 3.3 and Lemma 3.6, let  $\alpha_1 = \max(\alpha_*, \alpha^\dagger)$ . Then, for any  $\alpha \in (\alpha_1, 1)$  and for any  $\mu \in (\nu_*, \nu)$ , there exists  $C = C(\mu, \alpha) > 0$  such that*

$$\|\mathcal{S}_\alpha(t) (\mathbb{I} - \mathbb{P}_\alpha)\|_{\mathcal{B}(\mathcal{X})} \leq C e^{-\mu t} \quad \forall t \geq 0$$

where  $\mathbb{P}_\alpha$  is the projection operator over  $\text{Span}(F_\alpha)$  in  $\mathcal{X}$ . In other words, for any  $h_0 \in \mathcal{X}$  the solution  $h = h(t, v)$  (in the sense of semigroups) of the equation

$$\partial_t h = \mathcal{L}_\alpha(h)$$

satisfies

$$\|h(t) - c G_\alpha\|_{\mathcal{X}} \leq C \|h_0\| e^{-\mu t} \quad \text{for } t \geq 0,$$

where  $c := \int_{\mathbb{R}^3} h_0$ .

*Proof.* Denote by  $(\mathcal{U}_\alpha(t))_{t \geq 0}$  the semigroup in  $\mathcal{X}$  generated by  $\mathcal{B}_\alpha$ . Since  $\mathcal{L}_\alpha = \mathcal{A}_\alpha + \mathcal{B}_\alpha$ , it is well known from Duhamel's formula that

$$\mathcal{S}_\alpha(t) = \mathcal{U}_\alpha(t) + \int_0^t \mathcal{S}_\alpha(t-s) \mathcal{A}_\alpha \mathcal{U}_\alpha(s) ds.$$

Since  $\mathcal{A}_\alpha$  is weakly compact in  $\mathcal{X}$ , one gets that, for any  $t \geq s \geq 0$ , the integrand  $\mathcal{S}_\alpha(t-s) \mathcal{A}_\alpha \mathcal{U}_\alpha(s)$  is a weakly compact operator in  $\mathcal{X}$ . Using then the “strong compactness property” (see [9, Theorem C.7] for general reference and [19] for the extension to weakly compact operators in  $L^1$ -spaces) we get that

$$\mathcal{S}_\alpha(t) - \mathcal{U}_\alpha(t) \text{ is a weakly compact operator in } \mathcal{X} \text{ for all } t \geq 0.$$

We recall that, by definition, the Schechter essential spectrum is stable under compact perturbations. However, it can be shown that in  $L^1$ -spaces it is actually stable under *weakly compact* perturbations, see [14, Theorem 3.2 & Remark 3.3]. Due to this property we have

$$\mathfrak{S}_{\text{ess}}(\mathcal{S}_\alpha(t)) = \mathfrak{S}_{\text{ess}}(\mathcal{U}_\alpha(t)) \quad \forall t \geq 0.$$

In particular, the two  $C_0$ -semigroups share the same *essential type*, i.e.  $\omega_{\text{ess}}(\mathcal{S}_\alpha) = \omega_{\text{ess}}(\mathcal{U}_\alpha)$ . Since  $\omega_{\text{ess}}(\mathcal{U}_\alpha) \leq \omega_0(\mathcal{U}_\alpha)$  we get

$$\omega_{\text{ess}}(\mathcal{S}_\alpha) \leq -\beta_\star < 0.$$

Since

$$\omega_0(\mathcal{S}_\alpha) = \max(\omega_{\text{ess}}(\mathcal{S}_\alpha), s(\mathcal{L}_\alpha))$$

with  $s(\mathcal{L}_\alpha) = 0$  we obtain that

$$\omega_0(\mathcal{S}_\alpha) = 0 > \omega_{\text{ess}}(\mathcal{S}_\alpha).$$

General theory of  $C_0$ -semigroups [9, Theorem V.3.1, page 329] ensures that, for any  $\omega > \omega_{\text{ess}}(\mathcal{S}_\alpha)$ , one has

$$\mathfrak{S}(\mathcal{L}_\alpha) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq \omega\} = \{\lambda_1, \dots, \lambda_\ell\}$$

with  $\lambda_i$  eigenvalue of  $\mathcal{L}_\alpha$  with finite algebraic multiplicities  $k_i$  and  $\operatorname{Re} \lambda_1 \geq \dots \geq \operatorname{Re} \lambda_\ell$  and there is  $C_\omega > 0$  such that

$$\|\mathcal{S}_\alpha(t)(I - \Pi_\omega)\| \leq C_\omega \exp(\omega t) \quad \forall t \geq 0$$

where  $\Pi_\omega$  is the spectral projection associated to the set  $\{\lambda_1, \dots, \lambda_\ell\}$ . In particular, choosing  $\ell = 2$ , since  $\lambda_1 = 0$  while  $\operatorname{Re} \lambda_2$  is the spectral gap of  $\mathcal{L}_\alpha$  (see Theorem 3.3), choosing then  $0 < \mu < \nu_\star$ ,

$$\mathfrak{S}(\mathcal{L}_\alpha) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq -\mu\} = \{\lambda_1\} = \{0\}$$

we get the result since  $\Pi_\mu = \mathbb{P}_\alpha$  is the spectral projection on the simple eigenvalue 0.  $\square$

## 4. LOCAL STABILITY OF THE STEADY STATE

Using the result of the previous section, we show here that when  $\alpha$  is close to 1 and the initial condition is close to the steady state, solutions to Eq. (2.8) converge to it exponentially fast. Choosing  $\alpha$  close enough to 1 and the initial condition close enough to equilibrium, the exponential speed of convergence can be as close to  $\nu$  as we want. In all this section, we shall assume that the initial datum  $f_0$  is such that there exist  $b > 0, s \in (0, 1)$  satisfying

$$\int_{\mathbb{R}^3} f_0(v) \exp(b|v|) dv < \infty. \quad (4.1)$$

Then, according to Theorem 2.1, there exists  $0 < a < b$  (independent of  $\alpha$ ) such that the solution  $f(t)$  of (2.8) satisfies

$$f(t) \in \mathcal{X} \quad \forall t \geq 0$$

where we recall that

$$m(v) = \exp(-a|v|), \quad (a > 0), \quad \text{and} \quad \mathcal{X} = L^1(\mathbb{R}^3, m^{-1}(v) dv).$$

Actually, assuming a bit more on the initial datum, one can prove the following where we recall that  $\mathcal{Y} = L^1_1(m^{-1})$ :

**Lemma 4.1.** *Let  $f(t, v)$  be the solution to (2.8) associated to a nonnegative initial condition  $f_0$  satisfying (1.4) and (4.1). Then, for any  $\varepsilon > 0$  there exists  $C > 0$  depending only on  $\varepsilon$  and the moment (4.1) of  $f_0$  such that*

$$\|f(t)\|_{\mathcal{Y}} \leq C \|f(t)\|_{\mathcal{X}}^{1-\varepsilon} \quad \forall t \geq 0. \quad (4.2)$$

*Proof.* By Hölder's inequality, taking a dual pair  $p, q$ , i.e.  $1/p + 1/q = 1$

$$\begin{aligned} \|f(t)\|_{\mathcal{Y}} &= \int_{\mathbb{R}^3} f(t, v) \langle v \rangle m^{-1}(v) dv \\ &\leq \left( \int_{\mathbb{R}^3} f(t, v) m^{-1}(v) dv \right)^{1/p} \left( \int_{\mathbb{R}^3} f(t, v) \langle v \rangle^q m^{-1}(v) dv \right)^{1/q} \\ &= \|f(t)\|_{\mathcal{X}}^{1/p} \left( \int_{\mathbb{R}^3} f(t, v) \langle v \rangle^q m^{-1}(v) dv \right)^{1/q}. \end{aligned}$$

The initial datum  $f_0$  has an exponential tail of order 1 and Theorem 2.1 ensures that this property propagates with time; hence for any  $p \geq 1$  there exists  $C > 0$  such that

$$\sup_{t \geq 0} \left( \int_{\mathbb{R}^3} f(t, v) \langle v \rangle^q m^{-1}(v) dv \right)^{1/q} \leq C < \infty,$$

where the constant  $C > 0$  depends only on  $q$  and  $f_0$  (and not on  $\alpha$ ). Hence, choosing  $p$  such that  $1/p > 1 - \varepsilon$  proves the lemma.  $\square$

We deduce from the above lemma the following stability result:

**Theorem 4.2.** *Take  $0 < \nu_* < \nu$ . There exist  $0 < \alpha_* < 1$  and  $\varepsilon > 0$  such that, for all  $\alpha$  with  $\alpha_* \leq \alpha \leq 1$  and all nonnegative  $f_0 \in \mathcal{Y}$  satisfying (1.4) and (4.1) and such that  $\|f_0 - F_\alpha\|_{\mathcal{X}} \leq \varepsilon$ , the solution  $f(t)$  of (2.8) with initial data  $f_0$  satisfies*

$$\|f(t) - F_\alpha\|_{\mathcal{X}} \leq C \exp(-\nu_* t) \|f_0 - F_\alpha\|_{\mathcal{X}} \quad (t \geq 0) \quad (4.3)$$

for some constant  $C$  which depends only on  $\alpha$  and the moment (4.1) of the initial condition  $f_0$ .

*Proof.* Let  $\alpha \in (\alpha_0, 1]$  and let  $f(t, v)$  be the solution to (2.8) associated to the initial datum  $f_0$ . Setting  $h(t, v) := f(t, v) - F_\alpha(v)$ , we may rewrite (2.8) as

$$\partial_t h = \mathcal{L}_\alpha(h) + \mathcal{Q}_\alpha(h, h),$$

and through Duhamel's formula, denoting by  $(\mathcal{S}_\alpha(t))_{t \geq 0}$  the semigroup generated by  $\mathcal{L}_\alpha$ ,

$$h(t) = \mathcal{S}_\alpha(t)[h(0)] + \int_0^t \mathcal{S}_\alpha(t-s)[\mathcal{Q}_\alpha(h(s), h(s))] \, ds.$$

Take any  $\mu$  such that  $\nu_* < \mu < \nu$  and consider  $\alpha_0 < \alpha_* < 1$  such that for every  $\alpha_* \leq \alpha \leq 1$ , the operator  $\mathcal{L}_\alpha$  has a spectral gap of size  $\mu$  (the existence of such  $\alpha_*$  is warranted by Theorem 3.3). Then, for this range of  $\alpha$ , the semigroup  $((\mathbb{I} - \mathbb{P}_\alpha)\mathcal{S}_\alpha(t))_{t \geq 0}$  decays exponentially with speed  $\mu$ . Recalling that  $\text{Range}(\mathbb{I} - \mathbb{P}_\alpha) = \hat{\mathcal{X}}$  and that  $h(t) \in \hat{\mathcal{X}}$  for any  $t \geq 0$  we get that there exists  $C > 0$  which depends only on  $\alpha$  such that,

$$\begin{aligned} \|h(t)\|_{\hat{\mathcal{X}}} &\leq \|\mathcal{S}_\alpha(t)[h(0)]\|_{\hat{\mathcal{X}}} + \int_0^t \|\mathcal{S}_\alpha(t-s)[\mathcal{Q}_\alpha(h(s), h(s))]\|_{\hat{\mathcal{X}}} \, ds \\ &\leq C \|h(0)\|_{\hat{\mathcal{X}}} \exp(-\mu t) + C \int_0^t \|\mathcal{Q}_\alpha(h(s), h(s))\|_{\hat{\mathcal{X}}} \exp(-\mu(t-s)) \, ds \\ &\leq C \|h(0)\|_{\hat{\mathcal{X}}} \exp(-\mu t) + C \int_0^t \|h(s)\|_{\hat{\mathcal{Y}}}^2 \exp(-\mu(t-s)) \, ds \\ &\leq C \|h(0)\|_{\hat{\mathcal{X}}} \exp(-\mu t) + C_2 \int_0^t \|h(s)\|_{\hat{\mathcal{X}}}^{3/2} \exp(-\mu(t-s)) \, ds, \end{aligned}$$

where we have used Lemma 4.1 with  $\varepsilon = 1/4$ .

From this point, a Gronwall argument is enough to show that for  $\|h(0)\|_{\hat{\mathcal{X}}}$  small enough,  $h(t)$  converges exponentially fast to 0, with a speed as close to  $\mu$  as we want. Let us develop this argument more precisely. Take  $\delta > 0$  and  $f_0$  such that  $\|h(0)\|_{\hat{\mathcal{X}}} \leq \delta/2$ , and consider  $t$  in a time interval  $[0, T_\delta]$  where  $\|h(t)\|_{\hat{\mathcal{X}}} \leq \delta$ . Then,

$$\|h(t)\|_{\hat{\mathcal{X}}} \leq C \|h(0)\|_{\hat{\mathcal{X}}} \exp(-\mu t) + \delta^{1/2} C_2 \int_0^t \|h(s)\|_{\hat{\mathcal{X}}} \exp(-\mu(t-s)) \, ds$$

or, rewriting this for the quantity  $\gamma(t) := \exp(\mu t) \|h(t)\|_{\hat{\mathcal{X}}}$ ,

$$\gamma(t) \leq C \gamma(0) + \delta^{1/2} C_2 \int_0^t \gamma(s) \, ds \quad (t \in [0, T_\delta]).$$

Then, by Gronwall's Lemma,

$$\gamma(t) \leq C\gamma(0) \exp(\delta^{1/2}C_2t) \quad (t \in [0, T_\delta]),$$

or equivalently,

$$\|h(t)\|_{\hat{\mathcal{X}}} \leq C\|h(0)\|_{\hat{\mathcal{X}}} \exp((\delta^{1/2}C_2 - \mu)t) \quad (t \in [0, T_\delta]). \quad (4.4)$$

Now, take  $\delta$  small enough so that  $\delta^{1/2}C_2 - \mu < -\nu_*$ , and  $\varepsilon < \delta/2$  small so that

$$C\varepsilon \exp(-\nu_*t) \leq \delta.$$

Then, for  $\|h(0)\|_{\hat{\mathcal{X}}} < \varepsilon$ , one can actually take  $T_\delta = +\infty$  and (4.4) finishes the proof.  $\square$

## 5. GLOBAL STABILITY

**5.1. Evolution of the relative entropy.** We consider the evolution of the relative entropy of a solution  $f(t, v)$  to (1.1) with respect to the equilibrium Maxwellian  $\mathcal{M}$ . Notice that  $\mathcal{M}$  is *not the steady solution* associated to (1.1) and  $f(t, v)$  is not expected to converge towards  $\mathcal{M}$ . However, we shall take advantage of the entropy-entropy production estimate satisfied by  $\mathbf{L}$  to estimate first the distance from  $f(t)$  to  $\mathcal{M}$  which, combined with Theorem 2.5, yields a control of the distance between  $f(t)$  and  $F_\alpha$ . Using the entropy-entropy production estimate, Theorem 1.3, obtained in [5], one has the following crucial estimate on the evolution of the relative entropy along solutions to (1.1)

**Proposition 5.1.** *For any  $\alpha \in (\alpha_0, 1]$  and any nonnegative initial datum  $f_0$  with unit mass and  $H(f_0|\mathcal{M}) < \infty$ , the solution  $f(t) = f(t, v)$  to (1.1) satisfies:*

$$H(f(t)|\mathcal{M}) \leq \exp(-\lambda t)H(f_0|\mathcal{M}) + K(1 - \alpha) \quad \forall t \geq 0; \alpha \in (\alpha_0, 1]$$

for some positive constant  $K > 0$  independent of  $\alpha$  where  $\lambda > 0$  is the constant appearing in Theorem 1.3

*Proof.* Let  $\alpha \in (\alpha_0, 1]$  be fixed. Given a solution  $f(t) = f(t, v)$  to (1.1), set then

$$H(t) = H(f(t)|\mathcal{M}) = \int_{\mathbb{R}^3} f(t, v) \log \left( \frac{f(t, v)}{\mathcal{M}(v)} \right) dv.$$

Computing the time derivative of  $H(t)$  we get

$$\begin{aligned} \frac{d}{dt}H(t) &= \int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f) \log \left( \frac{f}{\mathcal{M}} \right) dv + \int_{\mathbb{R}^3} \mathbf{L}f \log \left( \frac{f}{\mathcal{M}} \right) dv \\ &= \int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f) \log f dv - \int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f) \log \mathcal{M} dv + \mathbf{D}(f). \end{aligned}$$

We recall [11] that

$$\int_{\mathbb{R}^3} \mathcal{Q}_\alpha(g, g)(v) \log g(v) dv = -\mathcal{D}_{H, \alpha}(g) + \frac{1 - \alpha^2}{2\alpha^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(v)g(w)|v - w| dv dw, \quad (5.1)$$

where the entropy production functional  $\mathcal{D}_{H,\alpha}(g)$  is defined, for any nonnegative  $g$ , by

$$\begin{aligned} \mathcal{D}_{H,\alpha}(g) &= \frac{1}{8\pi} \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} |v - w| g(v) g(w) \\ &\quad \times \left( \frac{g(v')g(w')}{g(v)g(w)} - \log \frac{g(v')g(w')}{g(v)g(w)} - 1 \right) d\sigma dv dw \geq 0 \end{aligned}$$

where the post-collisional velocities  $(v', w') = (v'_\alpha, w'_\alpha)$  are defined in (2.4). Moreover, using the definition of  $\mathcal{M}$  and the fact that  $\mathcal{Q}_\alpha$  conserves mass, one has

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f) \log \mathcal{M} dv &= -\frac{1}{2\Theta} \int_{\mathbb{R}^3} \mathcal{Q}_\alpha(f, f) |v|^2 dv \\ &= +\frac{1-\alpha^2}{16\Theta} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) |v - w|^3 dv dw \end{aligned} \quad (5.2)$$

where we used (2.2)-(2.3) noticing that  $\mathcal{A}_\alpha[|\cdot|^2](v, w) = -\frac{1-\alpha^2}{4} |v - w|^2$ . Putting (5.1) and (5.2) together we obtain

$$\begin{aligned} \frac{d}{dt} H(t) &= -\mathbf{D}(f(t)) - \mathcal{D}_{H,\alpha}(f(t)) + \frac{1-\alpha^2}{2\alpha^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) |v - w| dv dw \\ &\quad - \frac{1-\alpha^2}{16\Theta} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) |v - w|^3 dv dw \\ &\leq -\lambda H(t) + \frac{1-\alpha}{2\alpha_0^2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, v) f(t, w) |v - w| dv dw. \end{aligned}$$

Using the uniform control of moments of  $f$  one sees that there exists some positive constant  $C > 0$  independent of  $\alpha$  such that

$$\frac{d}{dt} H(t) \leq -\lambda H(t) + C(1-\alpha) \quad \forall t \geq 0, \forall \alpha \in (\alpha_0, 1].$$

Integrating this inequality, we obtain

$$H(f(t)|\mathcal{M}) \leq \exp(-\lambda t) H(f_0|\mathcal{M}) + \frac{C(1-\alpha)}{\lambda} (1 - \exp(-\lambda t))$$

which gives the result.  $\square$

**5.2. Global stability result.** Using the Csiszár-Kullback inequality we deduce from Proposition 5.1 the following

**Theorem 5.2.** *There exist some  $T = T(\alpha_0)$  and some function  $\ell : (\alpha_0, 1] \rightarrow \mathbb{R}^+$  with  $\lim_{\alpha \rightarrow 1} \ell(\alpha) = 0$  such that*

$$\|f(t) - F_\alpha\|_{\mathcal{X}} \leq \ell(\alpha) \quad \forall t \geq T(\alpha_0); \alpha \in (\alpha_0, 1].$$

We remark that both  $T$  and  $\ell$  in the above theorem can be given explicitly.

*Proof.* Using both the Csiszár-Kullback inequality and Holder's inequality we get

$$\begin{aligned} \|f(t) - \mathcal{M}\|_{L^1(m^{-1})} &\leq \|f(t) - \mathcal{M}\|_{L^1}^{1/2} \|f(t) - \mathcal{M}\|_{L^1(m^{-2})}^{1/2} \\ &\leq \sqrt{2} H(f(t)|\mathcal{M}) \|f(t) - \mathcal{M}\|_{L^1(m^{-2})}^{1/2}. \end{aligned}$$

Due to Theorem 2.1 (and recalling the definition of  $m$  by (4)), assuming  $a < r/2$  gives

$$\sup_{t \geq 0} \|f(t) - \mathcal{M}\|_{L^1(m^{-2})} < \infty.$$

Using now Proposition 5.1 we get the existence of two positive constants  $C_1, C_2 > 0$  independent of  $\alpha \in (\alpha_0, 1]$  such that

$$\|f(t) - \mathcal{M}\|_{\mathcal{X}} \leq C_1 \exp(-\lambda t) + C_2(1 - \alpha) \quad \forall t \geq 0; \alpha \in (\alpha_0, 1].$$

This, combined with Theorem 2.5, yields

$$\|f(t) - F_\alpha\|_{\mathcal{X}} \leq C_1 \exp(-\lambda t) + \eta_1(\alpha) \quad \forall t \geq 0; \alpha \in (\alpha_0, 1] \quad (5.3)$$

where  $\eta_1(\cdot)$  is a given explicit function with  $\lim_{\alpha \rightarrow 1} \eta_1(\alpha) = 0$ . We get readily the conclusion.  $\square$

The previous results essentially contain the proof of our main result:

*Proof of Theorem 1.2.* With the notation of Theorem 4.2, one can pick  $\alpha_1 \in (\alpha_0, 1)$  so that  $\ell(\alpha) \leq \varepsilon$  for any  $\alpha \in (\alpha_1, 1)$  so that

$$\|f(t) - F_\alpha\|_{\mathcal{X}} \leq \varepsilon \quad \forall t \geq T(\alpha_1) : \forall \alpha \in (\alpha_1, 1]$$

and Theorem 4.2 yields the global stability result.  $\square$

**Acknowledgments.** B. L. acknowledges support of the *de Castro Statistics Initiative*, Collegio C. Alberto, Moncalieri, Italy. J. A. C. was supported by the Marie-Curie CIG grant KineticCF and the Spanish project MTM2011-27739-C04-02.

#### APPENDIX A. SOME RESULTS IN PERTURBATION THEORY OF LINEAR OPERATORS

We gather here some results that are needed in Section 3.1 in order to study the spectral properties of  $\mathcal{L}_\alpha$  for  $\alpha$  close to 1. We begin by defining the *gap* between two closed linear operators, following [13, IV.2.4, p. 201]:

**Definition A.1.** Let  $X, Y$  be Banach spaces and  $S, T$  closed linear operators from  $X$  to  $Y$ . Let  $G(S), G(T)$  be their graphs, which are closed linear subspaces of  $X \times Y$ . We set

$$\delta(S, T) := \delta(G(S), G(T)) := \sup_{\substack{u \in G(S) \\ \|u\|_{X \times Y} = 1}} \text{dist}(u, G(T)),$$

and we define the *gap* between  $S$  and  $T$  as its symmetrisation:

$$\hat{\delta}(S, T) := \max\{\delta(S, T), \delta(T, S)\}.$$

We include here Theorem 2.14 from page 203 of [13] for the convenience of the reader:



**Theorem A.2** ([13, Thm. 2.14, p. 203]). *Let  $X, Y$  be Banach spaces and  $A, T$  be closed operators between  $X$  and  $Y$  such that  $A$  is  $T$ -bounded with relative bound less than one; that is,*

$$\|Au\|_Y \leq a\|u\|_X + b\|Tu\|_Y, \quad u \in \mathcal{D}(T).$$

*for some  $a \geq 0, 0 \leq b < 1$ . Then  $T + A$  is a closed operator and*

$$\hat{\delta}(S, T) \leq \frac{\sqrt{a^2 + b^2}}{1 - b}.$$

Finally, the following theorem is the main perturbation result we use in order to deduce the properties of the spectrum of  $\mathcal{L}_\alpha$ :

**Theorem A.3** ([13, Thm. 3.16, p. 212]). *Let  $T$  be a closed linear operator on a Banach space  $X$  and assume its spectrum  $\mathfrak{S}(T)$  is separated into two parts by a closed curve  $\Gamma$  in  $\mathbb{C}$ . Let  $X = X'_T \oplus X''_T$  be the associated decomposition of  $X$ . Then there exists  $\delta > 0$ , depending on  $T$  and  $\Gamma$ , such that any operator  $S$  on  $X$  with*

$$\hat{\delta}(S, T) < \delta$$

*satisfies the following properties:*

- (1) *The spectrum  $\mathfrak{S}(S)$  is also separated into two parts by the curve  $\Gamma$ .*
- (2) *In the associated decomposition  $X = X'_S \oplus X''_S$ , the spaces  $X'_S$  and  $X''_S$  are respectively isomorphic to  $X'_T$  and  $X''_T$ .*
- (3) *The decomposition  $X = X'_S \oplus X''_S$  is continuous in  $S$  in the sense that the projection  $P_S$  of  $X$  onto  $X'_S$  along  $X''_S$  tends to  $P_T$  in norm as  $\hat{\delta}(S, T) \rightarrow 0$ .*

## APPENDIX B. PROOF THAT $\mathcal{L}_\alpha$ GENERATES AN EVOLUTION SEMIGROUP

Let

$$m(v) = \exp(-a|v|), \quad a > 0$$

be fixed and let

$$\mathcal{X} = L^1(m^{-1}(v) dv), \quad \mathcal{Y} = L^1(\langle v \rangle m^{-1}(v) dv).$$

We wish here to investigate the compactness properties of the 'gain' part of  $\mathcal{L}_\alpha$  for  $\alpha \in (\alpha_0, 1]$ , with the final aim of showing that  $\mathcal{L}_\alpha$  generates a semigroup for all  $0 < \alpha \leq 1$ . Recall that

$$\mathcal{L}_\alpha h = \mathcal{Q}_\alpha(h, \mathbf{F}_\alpha) + \mathcal{Q}_\alpha(\mathbf{F}_\alpha, h) + \mathbf{L}(h), \quad h \in \mathcal{Y}.$$

and also that

$$\mathbf{L}(h) = \mathcal{K}h - \Sigma(\cdot)h$$

where

$$\mathcal{K}h(v) = \mathcal{Q}_1^+(h, \mathcal{M})(v) = \int_{\mathbb{R}^3} k(v, w)h(w) dw \quad \text{and} \quad \Sigma(v) = \int_{\mathbb{R}^3} \mathcal{M}(w)|v - w| dw$$

with

$$k(v, w) = C_0|v - w|^{-1} \exp \left\{ -\beta_0 \left( |v - w| + \frac{|v - \underline{u}|^2 - |w - \underline{u}|^2}{|v - w|} \right)^2 \right\} \quad (\text{B.1})$$

with  $\beta_0 = \frac{1}{8\Theta}$  and  $C_0 > 0$  a positive constant (depending only on  $\Theta_0$ ). Notice moreover that  $\Sigma(v) = \int_{\mathbb{R}^3} k(w, v) dw$ . In the same way, one can write

$$\mathcal{Q}_\alpha(h, \mathbf{F}_\alpha) = \mathcal{K}_\alpha^1 h - \sigma_\alpha(\cdot)h \quad \text{and} \quad \mathcal{Q}_\alpha(\mathbf{F}_\alpha, h) = \mathcal{K}_\alpha^2 h - \mathcal{K}_\alpha^3 h$$

with

$$\mathcal{K}_\alpha^1 h = \mathcal{Q}_\alpha^+(h, \mathbf{F}_\alpha), \quad \mathcal{K}_\alpha^2(h) = \mathcal{Q}_\alpha^+(\mathbf{F}_\alpha, h)$$

while

$$\sigma_\alpha(v) = \int_{\mathbb{R}^3} \mathbf{F}_\alpha(w) |v - w| dw \quad \text{and} \quad \mathcal{K}_\alpha^3(h)(v) = \mathbf{F}_\alpha(v) \int_{\mathbb{R}^3} h(w) |v - w| dw.$$

With this notation

$$\mathcal{L}_\alpha h = \mathcal{K}h + \mathcal{K}_\alpha^1 h + \mathcal{K}_\alpha^2 h - (\Sigma + \sigma_\alpha)h - \mathcal{K}_\alpha^3 h \quad \forall h \in \mathcal{Y}.$$

As for  $\mathbf{L}$ , the two operators  $\mathcal{K}_\alpha^i$ ,  $i = 1, 2$  are integral operators with explicit kernels. Namely,

**Lemma B.1.** *For any  $h \in \mathcal{Y}$ , one has*

$$\mathcal{K}_\alpha^1 h(v) = \int_{\mathbb{R}^3} K_\alpha^1(v, w) h(w) dw$$

where

$$K_\alpha^1(v, w) = \frac{C_\alpha}{|v - w|} \int_{V_2 \cdot (w - v) = 0} \mathbf{F}_\alpha \left( v + V_2 + \frac{\alpha - 1}{\alpha + 1} (w - v) \right) dV_2 \quad (\text{B.2})$$

for some positive constant  $C_\alpha > 0$ .

*Proof.* The proof follows standard computations performed for instance in [2] where  $\mathbf{F}_\alpha$  was replaced by a given Maxwellian. In particular, (B.2) is derived in [2, p. 524].  $\square$

Recalling that there exist two Maxwellian distributions  $\underline{\mathcal{M}}$  and  $\overline{\mathcal{M}}$  (independent of  $\alpha$ ) such that

$$\underline{\mathcal{M}}(v) \leq F_\alpha(v) \leq \overline{\mathcal{M}}(v) \quad \forall v \in \mathbb{R}^3, \quad \forall \alpha \in (0, 1).$$

In particular, this proves that, for any  $h \geq 0$ ,

$$\mathcal{K}_\alpha^1 h \leq \overline{\mathcal{K}}_\alpha^1 h = \mathcal{Q}_\alpha^+(h, \overline{\mathcal{M}})$$

and

$$\mathcal{K}_\alpha^2 h \leq \overline{\mathcal{K}}_\alpha^2 h = \mathcal{Q}_\alpha^+(\overline{\mathcal{M}}, h).$$

Again,  $\overline{\mathcal{K}}_\alpha^1 h$  is an integral kernel with explicit kernel, namely

$$\overline{\mathcal{K}}_\alpha^1 h(v) = \int_{\mathbb{R}^3} \overline{K}_\alpha^1(v, w) h(w) dw$$

with

$$\overline{K}_\alpha^1(v, w) = \overline{C}_\alpha |v - w|^{-1} \exp \left\{ -\beta_1 \left( (1 + \mu_\alpha) |v - w| + \frac{|v - u_1|^2 - |w - u_1|^2}{|v - w|} \right)^2 \right\} \quad (\text{B.3})$$

where  $\overline{C}_\alpha > 0$  is a positive constant depending only on  $\alpha$  and  $\overline{\mathcal{M}}$  while

$$\mu_\alpha = 2\frac{1-\alpha}{1+\alpha} \geq 0, \quad \beta_1 = \frac{1}{8\Theta_1};$$

$\Theta_1, u_1$  being the kinetic energy and momentum of  $\overline{\mathcal{M}}$  (see [2]). By a simple domination argument (namely, Dunford-Pettis criterion), if  $\overline{K}_\alpha^i$  are weakly compact in  $\mathcal{X}$  then so will be  $K_\alpha^i, i = 1, 2$ .

**Proposition B.2.** *Let  $\alpha \in (\alpha_0, 1)$  be fixed. Then,*

$$\mathcal{K} : \mathcal{Y} \rightarrow \mathcal{X}, \quad \mathcal{K}_\alpha^1 : \mathcal{Y} \rightarrow \mathcal{X}$$

*are positive, bounded, weakly compact operators. Moreover,  $\mathcal{K}_\alpha^2 \in \mathcal{B}(\mathcal{X})$  while*

$$\mathcal{K}_\alpha^3 : \mathcal{X} \rightarrow \mathcal{X}$$

*is a bounded and weakly compact operator.*

*Proof.* The fact that  $\mathcal{K}, \mathcal{K}_\alpha^1$  are bounded operator from  $\mathcal{Y}$  to  $\mathcal{X}$  comes from Prop. 2.3. We divide the proof of the compactness properties into several steps.

*First step: weak compactness of  $\mathcal{K}_\alpha^1$ .* We already notice that it is enough to prove that  $\overline{\mathcal{K}}_\alpha^1 : \mathcal{Y} \rightarrow \mathcal{X}$  is weakly compact. Let  $d\nu(v) = m^{-1}dv$  and let  $\mathcal{B} = B_{\mathcal{Y}}$  be the unit ball of  $\mathcal{Y}$ . Since  $\mathcal{X} = L^1(\mathbb{R}^3, d\nu)$ , according to Dunford-Pettis Theorem, this amounts to prove that

$$\sup_{h \in \mathcal{B}} \int_A |\overline{\mathcal{K}}_\alpha^1 h(v)| d\nu(v) \longrightarrow 0 \quad \text{as} \quad \nu(A) \rightarrow 0 \quad (\text{B.4})$$

and

$$\sup_{h \in \mathcal{B}} \int_{|v-u_1|>r} |\overline{\mathcal{K}}_\alpha^1 h(v)| d\nu(v) \longrightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \quad (\text{B.5})$$

Using the representation of  $\overline{\mathcal{K}}_\alpha^1$  as an integral operator, it is easy to check that (B.4) and (B.5) will follow if one is able prove that

$$\sup_{w \in \mathbb{R}^3} \frac{m(w)}{\langle w \rangle} \int_A \overline{K}_\alpha^1(v, w) m^{-1}(v) dv \longrightarrow 0 \quad \text{as} \quad \nu(A) \rightarrow 0 \quad (\text{B.6})$$

and

$$\sup_{w \in \mathbb{R}^3} \frac{m(w)}{\langle w \rangle} \int_{|v-u_1|>r} \overline{K}_\alpha^1(v, w) m^{-1}(v) dv \longrightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \quad (\text{B.7})$$

Let us prove (B.6). Let  $A \subset \mathbb{R}^3$  be a given Borel subset and let  $w \in \mathbb{R}^3$  be fixed. Set  $B_w = \{v \in \mathbb{R}^3, |v - w| < 1\}$ . Since  $\overline{K}_\alpha^1(v, w) \leq \overline{C}_\alpha |v - w|^{-1}$  one has

$$\begin{aligned} \int_A \overline{K}_\alpha^1(v, w) m^{-1}(v) dv &\leq \overline{C}_\alpha \int_A |v - w|^{-1} d\nu(v) \\ &= \overline{C}_\alpha \left( \int_{A \cap B_w} |v - w|^{-1} d\nu(v) + \int_{A \cap B_w^c} |v - w|^{-1} d\nu(v) \right). \end{aligned}$$

Clearly

$$\int_{A \cap B_w^c} |v - w|^{-1} d\nu(v) \leq \nu(A)$$

while, for any  $p > 1$ ,  $1/q + 1/p = 1$ , one has

$$\begin{aligned} \int_{A \cap B_w} |v - w|^{-1} d\nu(v) &\leq \left( \int_{A \cap B_w} d\nu(v) \right)^{1/q} \left( \int_{A \cap B_w} |v - w|^{-p} \exp(a|v|) dv \right)^{1/p} \\ &\leq \exp\left(\frac{a}{p}(|w| + 1)\right) \nu(A)^{1/q} \left( \int_{B_w} |v - w|^{-p} dv \right)^{1/p} \end{aligned}$$

where we used that,  $\exp(a|v|) \leq \exp(a|w|) \exp(a|v - w|)$  for any  $v, w$ . Choosing now  $p > 3$ , one sees that

$$\left( \int_{B_w} |v - w|^{-p} dv \right)^{1/p} < \infty$$

and is independent of  $w$ . Thus, there exists  $C = C(\alpha, a, p)$  such that

$$\int_A \bar{K}_\alpha^1(v, w) m^{-1}(v) dv \leq C \left( \exp\left(\frac{a}{p}|w|\right) \nu(A)^{1/q} + \nu(A) \right) \quad \forall w \in \mathbb{R}^3.$$

Since  $p > 1$ , this proves that (B.6) holds true. Let us now prove (B.7). One first notice that

$$\sup_{|w - u_1| \leq r/2} \frac{m(w)}{\langle w \rangle} \int_{|v - u_1| > r} \bar{K}_\alpha^1(v, w) m^{-1}(v) dv \longrightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (\text{B.8})$$

Indeed, one notices that

$$\bar{K}_\alpha^1(v, w) \leq \bar{C}_\alpha |v - w|^{-1} \exp\left(2\beta_1(1 + \mu_\alpha) [|w - u_1|^2 - |v - u_1|^2]\right).$$

Therefore, if  $|w - u_1| \leq r/2$  and  $|v - u_1| > r$ , one gets

$$\bar{K}_\alpha^1(v, w) \leq \frac{2\bar{C}_\alpha}{r} \exp\left(-\frac{3}{2}\beta_1(1 + \mu_\alpha)|v - u_1|^2\right)$$

and (B.8) follows easily since

$$\int_{\mathbb{R}^3} \exp\left(-\frac{3}{2}\beta_1(1 + \mu_\alpha)|v - u_1|^2\right) d\nu(v) < \infty.$$

Now, to prove (B.7), it is enough to show that

$$\sup_{|w - u_1| > r/2} \frac{m(w)}{\langle w \rangle} \int_{|v - u_1| > r} \bar{K}_\alpha^1(v, w) m^{-1}(v) dv \longrightarrow 0 \quad \text{as } r \rightarrow \infty. \quad (\text{B.9})$$

Arguing as in [4, Proposition A.1] (with  $s = 1$ ), there exists  $K = K_\alpha > 0$  such that

$$\int_{\mathbb{R}^3} \bar{K}_\alpha^1(v, w) m^{-1}(v) dv \leq K_\alpha m^{-1}(w).$$

Therefore, for any  $r > 0$ ,

$$\frac{m(w)}{\langle w \rangle} \int_{|v - u_1| > r} \bar{K}_\alpha^1(v, w) m^{-1}(v) dv \leq K_\alpha \langle w \rangle^{-1}$$

and (B.9) follows since  $\sup_{|w-u_1|>r/2} \langle w \rangle^{-1} \rightarrow 0$  as  $r \rightarrow \infty$ . This achieves to prove that  $\overline{K}_\alpha^1 : \mathcal{Y} \rightarrow \mathcal{X}$  is weakly compact.

*Second step: weak compactness of  $\mathcal{K}$ .* Notice that  $\mathcal{K}$  and  $\overline{\mathcal{K}}_\alpha^1$  are two integral operators whose kernels, given respectively by (B.1) and (B.3), are very similar. The above computations can then be reproduced *mutatis mutandis* to get the weak-compactness of  $\mathcal{K}$ .

*Third step: boundedness of  $\mathcal{K}_\alpha^2$ .* According to [1, Theorem 12], and since  $m(v) = \exp(-a|v|^s)$  with  $s = 1$ , one has

$$\overline{\mathcal{K}}_\alpha^2 = \mathcal{Q}_\alpha^+(\overline{\mathcal{M}}, \cdot) : \mathcal{X} \rightarrow \mathcal{X}$$

is bounded for any  $0 < \alpha < 1$ . Consequently, simple domination argument asserts that  $\mathcal{K}_\alpha^2 \in \mathcal{B}(\mathcal{X})$ .

*Final step: weak compactness of  $\mathcal{K}_\alpha^3$ .* Recall that

$$\mathcal{K}_\alpha^3 h(v) = \mathbf{F}_\alpha(v) \int_{\mathbb{R}^3} h(w) |v - w| dw.$$

Therefore,

$$|\mathcal{K}_\alpha^3 h(v)| \leq \langle v \rangle \mathbf{F}_\alpha(v) \int_{\mathbb{R}^3} |h(w)| \langle w \rangle dw.$$

In particular, there exists  $C > 0$  such that

$$|\mathcal{K}_\alpha^3 h(v)| \leq \langle v \rangle \mathbf{F}_\alpha(v) \int_{\mathbb{R}^3} |h(w)| m^{-1}(w) dw.$$

Since

$$\int_{\mathbb{R}^3} \mathbf{F}_\alpha(v) \langle v \rangle m^{-1}(v) dv < \infty$$

this proves that  $\mathcal{K}_\alpha^3 : \mathcal{X} \rightarrow \mathcal{X}$  is bounded and dominated by a one-rank operator. In particular, it is weakly compact.  $\square$

As a general consequence, one has the following

**Theorem B.3.** *For any  $\alpha \in (\alpha_0, 1)$ , the unbounded operator  $\mathcal{L}_\alpha$  is the generator of a  $C_0$ -semigroup  $(\mathcal{S}_\alpha(t))_{t \geq 0}$  in  $\mathcal{X}$ .*

*Proof.* One applies the recent version of Desch theorem for positive semigroups in  $L^1$ -spaces, see for instance [20]. For any  $\alpha \in (\alpha_0, 1)$ , define the multiplication operator:

$$A_\alpha h(v) = -(\sigma_\alpha(v) + \Sigma(v))h(v), \quad h \in \mathcal{D}(A_\alpha) = \mathcal{Y}$$

then,  $A_\alpha$  is the generator of a positive  $C_0$ -semigroup  $(U_\alpha(t))_{t \geq 0}$  in  $\mathcal{X}$ . Since  $\mathcal{K}$  and  $\mathcal{K}_\alpha^1$  are weakly compact, one has from [20] that  $B_\alpha = A_\alpha + \mathcal{K} + \mathcal{K}_\alpha^1$  is the generator of a positive  $C_0$ -semigroup  $(V_\alpha(t))_{t \geq 0}$ . Finally, since  $\mathcal{K}_\alpha^2$  and  $\mathcal{K}_\alpha^3$  are bounded operators, one gets that

$$\mathcal{L}_\alpha = B_\alpha + \mathcal{K}_\alpha^2 - \mathcal{K}_\alpha^3$$

is the generator of a  $C_0$ -semigroup in  $\mathcal{X}$ .  $\square$

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